39. A Proposition on the Cardinality of Closed Discrete Subsets of a Topological Space

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D. B. Gauld and M. K. Vamanamurthy [3] have considered on a problem of the cardinality of a closed discrete subset of a separable normal space. In this paper, we prove that the cardinality of each closed discrete subset of a separable normal space being at most countable is independent of the usual axioms of set theory, i.e. ZFC.

Theorem 1. *If X is a normal space, then the cardinality of each closed discrete subspace of X is less than the exponential of its density.*

Proof. Let D be a dense subset of X having the cardinality of the density of X. Assume C is a closed discrete subspace of X, moreover A is a subset of C. Since C is closed discrete, both A and $C-A$ are closed subsets of X. Then by the normality of X, there exist disjoint open sets U_A and V_A in X such that A is contained in U_A and $C-A$ is contained in V_A. Next define a mapping f from the power set of C to the power set of D such that for each subset A of C, $f(A) = D \cap U_A$. Then clearly f is a one-to-one mapping. Hence $\exp|C| \leqq \exp|D|$. Therefore, $|C| < \exp|D|$. We complete the proof.

Remark. In general, it is not always $\kappa \leqq \lambda$, whenever $\exp\kappa \leqq \exp\lambda$. For example, if Martin's axiom and the negation of the continuum hypothesis are assumed, then $\exp\aleph_0 = \exp\aleph_1$, but $\aleph_0 < \aleph_1$. Hence we can construct a separable normal space having uncountable closed discrete subsets assuming Martin's axiom and the negation of the continuum hypothesis. See Theorem 2.

Corollary 1. *If the generalized continuum hypothesis is assumed, then the cardinality of each closed discrete subset of a normal space is less than or equal to its density.*

Proof. Let κ be the density of a normal space X. Then $\exp\kappa$ equal to κ^+ by the assumption. Hence the cardinality of each closed discrete subset of X is less than or equal to κ by Theorem 1. We complete the proof.

We can prove the next result in a similar way to Corollary 1.

Corollary 2. *If the continuum hypothesis is assumed, then the cardinality of each closed discrete subset of a separable normal space is countable.*

Corollary 3. *The Sorgenfrey's square S is not normal.*

Proof. Let C be the subset $\{(x, y) \in S : y = -x\}$. Then C is a closed discrete subset of S with the cardinality of $\exp \aleph_0$. Clearly S is separable. If X is normal, then $|C| < \exp \aleph_0$ by Theorem 1. Hence a contradiction!

Next we construct an example of a separable normal space having an uncountable closed discrete subsets assuming Martin's axiom and the negation of the continuum hypothesis. First we need some well known results.

Lemma 1 (R. H. Bing [2]). *If there is an uncountable subset X of reals such that in the subspace topology, every subset of X is an G_δ, then $M(X) = \{(x, y) \in R \times R : y > 0\} \cup X \times \{0\}$ is normal. The topology of $M(X)$ defined as follows. A neighborhood of a point p of $X \times \{0\}$ consists of $\{p\}$ together with the interior of a circle tangent to the axis at p, and other points in the space receive the usual topology inherited from the plane.*

The following result is well known, see [1, 5.8].

Lemma 2. *Let Martin's axiom and the negation of the continuum hypothesis be assumed. If X is a subset of reals with the cardinality less than $\exp \aleph_0$, then every subset of X is G_δ in the subspace topology.*

Remark. We can easily show $M(X)$ is separable and $X \times \{0\}$ is closed discrete in $M(X)$. So we can show the next result by Lemmas 1 and 2.

Theorem 2. *Let Martin's axiom and the negation of the continuum hypothesis be assumed. If X is a subset of the reals such that $\aleph_0 < |X| < \exp \aleph_0$, then $X \times \{0\}$ is uncountable closed discrete subset of the separable normal space $M(X)$.*

Remark. Thus, we can show that the property of the cardinality of each closed discrete subset of a separable normal space being countable is independent of the usual axioms of set theory because of Corollary 2 and Theorem 2.

References

[1] H. R. Bennett and T. G. Mclaughlin: A selective survey of axiom sensitive results in general topology. Texas Tech. University Mathematics series, no. 12 (1976).

[2] R. H. Bing: Metrization of topological spaces. Can. J. Math., **3**, 175–186 (1951).

[3] D. B. Gauld and M. K. Vamanamurthy: Cardinality of discrete subsets of a topological space. Bull. Austral. Math. Soc., **25**, 99–101 (1982).