38. *The Existence of an Invariant Stable Foliation and the Problem of Reducing to a One-Dimensional Map*

By Michiko YURI
Department of Mathematics, Tsuda College

(Communicated by Kôsaku YOSIDA, M. J. A., April 12, 1983)

Introduction. In 1962, E. Lorenz found the first example of "strange-attractor" by investigating a hydrodynamical system. Recently another equation which is even simpler than the Lorenz equation has been proposed by O. E. Rössler, and by numerical solution, it was shown that these equations indicate the existence of a two-dimensional attractor which has a compact "ribbon-like" structure. Since the attractor can be treated as a "single-sheeted" quasi-two-dimensional object, we take a cut across the attractor and construct a Poincaré map by means of which we can reduce a three-dimensional continuous flow to a one-dimensional discrete map. Thus one-dimensional models serve as the simplest example of models for some dynamical system. They appear in the original paper by Lorenz, and also in more recent work by Guckenheimer, Rössler, and others [1]–[6].

This procedure, however, has not been justified rigorously so far. To be more precise with the problem, let us consider a two-dimensional map under some conditions;

H; $\mathrm{R}^2\to\mathrm{R}^2$, $H(x, y)=(f(x)+\varepsilon_1(x, y), \mu y+\varepsilon_2(x, y))$, where $\varepsilon_i(x, y)$ $(i=1, 2)$ is of class C^2, and $f:\mathrm{R}\to\mathrm{R}$ is a one-dimensional continuous map of piecewise C^2-class such that $f(I)\subset I$ (here I denotes $[0, 1]$).

This map could have an ω-limit set Ω near $I\times\{0\}$ under some conditions on $\varepsilon_i(x, y)$ and μ. So, if we could construct an invariant stable foliation on Ω, then we could say that the study of the behavior of the system near Ω can be reduced to the study of the one-dimensional map on Ω. One of the aims of this work is to show the existence of an invariant stable foliation on Ω almost everywhere with respect to Lebesgue measure. (For the precise definition of the invariant stable foliation, see Theorem 2.)

For instance, when $\varepsilon_1(x, y)=0$, $\varepsilon_2(x, y)=\hat{\varepsilon}x$, the map H has "trivial" invariant stable foliation which is $\{x=Constant\}$, and hence the behavior near the ω-limit set is reduced to the one-dimensional map f on I.

In this paper, we assume the following conditions on $\varepsilon_i(x, y)$

$(i=1,2)$ and μ; $\varepsilon_i(x,0)=0$, $|(\partial\varepsilon_i/\partial y)(x,0)|<\hat{\varepsilon}_i$ $(i=1,2)$ for $(x,0)\in I\times\{0\}$, and $\mu>\hat{\varepsilon}_2$, $\mu+\hat{\varepsilon}_1+\hat{\varepsilon}_2<1$.

Namely, H is supposed to leave invariant the set $I\times\{0\}\subset\mathrm{R}^2$. The tangent map $DH(x,0)$ at $(x,0)\in I\times\{0\}$ is expressed by an upper triangular matrix.

§ 1. The existence of an invariant stable foliation for a two-dimensional map. Recently, D. Ruelle has proved that if H is a diffeomorphism of a compact manifold, an invariant stable foliation exists almost everywhere with respect to H-invariant measure, meanwhile Pesin presented a stable manifold theorem under the assumption that a smooth invariant measure exists [7], [8]. However, since in our case H does not necessarily have an invariant measure which is absolutely continuous with respect to Lebesgue measure, we can not apply their results to our problem directly.

The proof of Ruelle's stable manifold theorem is based on the study of random matrix products and perturbations of such products occurring in the multiplicative ergodic theorem; that is, he used this theorem essentially [9]. In contrast, since in our problem the tangent map on $I\times\{0\}$ is expressed by an upper triangular matrix, this property makes it possible for us to form an invariant stable foliation without using the multiplicative ergodic theorem. We only require some assumption on the ratio of eigenvalues of the tangent maps.

Denote by $B(z,\alpha)$ the open ball of radius α centered at z in R^2, and by $\overline{B(z,\alpha)}$ its closure.

Theorem 1. *Let $H:\mathrm{R}^2\to\mathrm{R}^2$ be a differentiable map of C^2-class. Assume that there is a set $\Gamma\subset\mathrm{R}^2$ such that $H(\Gamma)\subset\Gamma$ and that a tangent map $DH(z)$ is expressed by an upper triangular matrix for all $z\in\Gamma$. We write for $z\in\Gamma$,*

$$T_n=DH(H^{n-1}(z))=\begin{pmatrix}\tilde{\alpha}_n & \tilde{\gamma}_n\\ 0 & \tilde{\beta}_n\end{pmatrix},\quad \textit{and}\quad T^n=T_n\circ\cdots\circ T_1=\begin{pmatrix}\alpha_n & \gamma_n\\ 0 & \beta_n\end{pmatrix}.$$

Suppose the following:

(1) $\limsup 1/n\log\|T_n\|\leqq 0$,

(2) *there exists* $\xi>0$ *such that* $|\beta_n/\alpha_{n+1}|<\exp(-n\xi)$ *for all* $n>0$,

(3) $\tilde{\xi}\equiv\limsup 1/n\log|\alpha_n/\beta_n|>\xi$,

(4) $\det T_n\neq 0$ *for all* $n>0$,

(5) $\limsup 1/n\log|\det T_n|=0$,

(6) $\hat{\beta}\equiv\limsup 1/n\log|\beta_n|<0$ *and* $5(\tilde{\xi}-\xi)<-\hat{\beta}$.

Let λ be a constant such that $-4(\tilde{\xi}-\xi)>\lambda>\hat{\beta}+(\tilde{\xi}-\xi)$.

Under these conditions, there are $\alpha_z>0$, $\gamma_z>0$ and $\Pi_z>0$ with the following properties:

(i) $S(\Pi_z)=\{u\in\overline{B(z,\alpha_z)}:\|H^n(z+u)-H^n(z)\|\leqq\Pi_z\cdot\exp(n\lambda)$ *for all* $n\geqq 0\}$ *is a C^1-submanifold of $\overline{B(z,\alpha_z)}$, tangent at z to U_z.*

(ii) *If* $u, v \in S(\Pi_z)$, *then* $\|H^n(z+u)-H^n(z+v)\| \leqq \gamma_z \|u-v\| \exp(n\lambda)$ *for all* $n \geqq 0$. (Here U_n denotes the corresponding eigenspace to the smallest eigenvalue of the matrix $\sqrt{{}^tT^n T^n}$ and $U_z = \lim_n U_n$.)

§ 2. **An application for our problem.** Now, let us apply Theorem 1 to our map $H: \mathrm{R}^2 \to \mathrm{R}^2$, $H(x, y) = (f(x) + \varepsilon_1(x, y),\ \mu y + \varepsilon_2(x, y))$, which is mentioned in the introduction. By using this theorem, we can obtain conditions on H which guarantee the existence of an invariant stable foliation almost everywhere with respect to Lebesgue measure, and furthermore these conditions can be expressed in terms of μ which measures a degree of contraction, perturbing terms ε_i and a one-dimensional map f. Our main result is the following.

Theorem 2.*) (1) *If f is a continuous map of piecewise* C^2*-class and* $\inf_x |f'_\pm(x)| > 0$, *and* μ, $\hat{\varepsilon}_i (i=1, 2)$ *are sufficiently small*

$$\left(\text{i.e. } \inf_x |f'_\pm(x)| > \mu + \hat{\varepsilon}_2,\ \left\{\frac{\sup_x |f'_\pm(x)|}{\inf_x |f'_\pm(x)|}\right\}^5 < \frac{(\mu - \hat{\varepsilon}_2)^5}{(\mu + \hat{\varepsilon}_2)^6}\right),$$

then the map H has an invariant stable foliation on $I \times \{0\}$. *That is, there exists* $\{C_x ; x \in I\}$ *such that*

(i) C_x *are Lipschitz-continuous curves,*

(ii) $(x, 0) \in C_x$,

(iii) $H(C_x) \subset C_{f(x)}$,

(iv) *there are* $\lambda < 0$ *and* $\gamma_x > 0$ *such that for* $z_1, z_2 \in C_x$

$$\|H^n(z_1) - H^n(z_2)\| \leqq \gamma_x \exp(n\lambda) \|z_1 - z_2\| \qquad (n \geqq 0).$$

(2) *Let* $f(x) = A \cdot x(1-x)$, *where* $0 < A \leqq 4$. *If f has a stable periodic point* $x_0 \in I \backslash \Lambda$, *where* Λ *denotes the set* $\{x \in I ; f^k(x) = 1/2$ *for some* $k \geqq 0\}$, *and* μ, $\hat{\varepsilon}_i$ *are sufficiently small, then the map H has an invariant stable foliation on* $I \times \{0\}$ *almost everywhere with respect to Lebesgue measure whose leaves are class* C^1.

The proof of Theorem 2 will be published in Tokyo Journal of Mathematics.

We remark that the conditions on f can be weakened considerably at the cost of the simplicity of their expression.

In conclusion, I would like to thank Profs. Toshio Niwa, Yôichirô Takahashi, Shunji Itô, and Shigehiro Ushiki for their encouragement and valuable advices.

References

[1] E. N. Lorenz: Deterministic nonperiodic flow. J. Atmos. Sci., **20**, 130–141 (1963).

[2] I. Shimada and T. Nagashima: On the *C*-system-like property of the Lorenz system. Progr. Theor. Phys., **58**, 1318–1319 (1978).

*) Added to this note; recently, the author has obtained the result which partially extends Theorem 2 by different method [10].

[3] O. E. Rössler: An Equation for continuous chaos. Phys. Lett., **57**A, 397–398 (1976).
[4] R. M. May: Simple mathematical models with very complicated dynamics. Nature, **261**, 459–467 (1976).
[5] J. Guckenheimer, G. Oster, and A. Ipaktchi: The dynamics of density dependent population models. J. Math. Biol., **4**, 101–147 (1977).
[6] R. Shaw: Strange attractors. Chaotic behavior and information flow. Z. Naturforsch., **36**a, 80–112 (1981).
[7] D. Ruelle: Ergodic theory of differentiable dynamical system. Publ. Math. Inst. HES, **50**, 27–58 (1981).
[8] Ya. B. Pesin: Lyapunov characteristic exponents and smooth ergodic theory. Uspehi Mat. Nauk, **32**, n4 (196), 55–112 (1977).
[9] V. I. Oseledec: A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical system. Trudy Mosk. Mat. Obsc., **19**, 179–231 (1968).
[10] M. Yuri: A construction of an invariant stable foliation by the shadowing lemma. Tokyo Journal of Math. (preprint).