37. *Invariant Measures on Orbits Associated to a Symmetric Pair**)

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§1. **Introduction.** Let $\mathfrak{g}_0$ be a real simple Lie algebra of non-compact type and let σ be an involution of $\mathfrak{g}_0$. Then by setting $\mathfrak{h}_0=\{X\in\mathfrak{g}_0;\sigma X=X\}$ and $\mathfrak{q}_0=\{X\in\mathfrak{g}_0;\sigma X=-X\}$, we obtain a direct sum decomposition $\mathfrak{g}_0=\mathfrak{h}_0+\mathfrak{q}_0$. We remark that M. Berger [1] gives the classification of such a pair $(\mathfrak{g}_0,\mathfrak{h}_0)$ which we call a symmetric pair in this paper. Let G be the adjoint group of $\mathfrak{g}_0$ and let H be the analytic subgroup of G corresponding to $\mathfrak{h}_0$. Since $[\mathfrak{h}_0,\mathfrak{q}_0]\subseteqq\mathfrak{q}_0$, H acts on $\mathfrak{q}_0$ by the adjoint action. For brevity, we set $h\cdot X=Ad_G(h)X$ for $h\in H$ and $X\in\mathfrak{q}_0$. If $\mathcal{O}(X)$ is the H-orbit of X, then $\mathcal{O}(X)$ is identified with the homogeneous space H/H_X, where $H_X=\{h\in H:h\cdot X=X\}$. We ask whether $\mathcal{O}(X)$ has an H-invariant measure or not. As was pointed out by van Dijk [5], every orbit does not have an H-invariant measure. Now assuming that the orbit $\mathcal{O}(X)$ of an element X of $\mathfrak{q}_0$ has an H-invariant measure $d\mu$, we define a functional T by

$$T(f)=\int_{\mathcal{O}(X)}fd\mu \qquad \text{for any } f\in C_0^\infty(\mathfrak{q}_0).$$

Then we next ask whether T defines a Radon measure on $\mathfrak{q}_0$ or not. Also, this does not hold in general even if the assumption on the existence of an H-invariant measure is satisfied.

In this note, we always assume that the complexification of $\mathfrak{g}_0$ is of type A and give a complete answer to the problem on the existence of H-invariant measures in §2 and discuss on the possibility of extending the measures to $\mathfrak{q}_0$ as H-invariant distributions in §3.

§2. **The existence of H-invariant measures.** We use the notation in the introduction. The complexifications of $\mathfrak{g}_0$, $\mathfrak{h}_0$, $\mathfrak{q}_0$ are denoted by $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{q}$, respectively.

In this note we always assume that $\mathfrak{g}$ is simple of type A. Then it follows from [1] that $(\mathfrak{g},\mathfrak{h})$ is isomorphic to one of the pairs $(\mathfrak{sl}(n,\boldsymbol{C}),\mathfrak{so}(n,\boldsymbol{C}))$, $(\mathfrak{sl}(2n,\boldsymbol{C}),\mathfrak{sp}(n,\boldsymbol{C}))$, $(\mathfrak{sl}(m+n,\boldsymbol{C}),\mathfrak{sl}(m,\boldsymbol{C})+\mathfrak{sl}(n,\boldsymbol{C})+\boldsymbol{C})$ $(m\geqq n\geqq 1)$.

We take an element X of $\mathfrak{q}_0$ and denote by $\mathfrak{h}_0(X)$ the Lie algebra of H_X, by $\mathfrak{h}(X)$ the complexification of $\mathfrak{h}_0(X)$. The following proposition is well-known but plays a fundamental role in the subsequent discussion.

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Proposition 1. *For an element X of $\mathfrak{q}_0$, the H-orbit $\mathcal{O}(X)$ has an H-invariant measure if and only if the Lie algebra $\mathfrak{h}(X)$ satisfies the condition*

(1) $\operatorname{tr} ad_{\mathfrak{h}(X)}(Z)=0$ *for any* $Z \in \mathfrak{h}(X)$.

From Proposition 1, we easily obtain

Theorem 2. *Let $(\mathfrak{g}_0, \mathfrak{h}_0)$ be a symmetric pair such that $(\mathfrak{g}, \mathfrak{h})$ is isomorphic to $(\mathfrak{sl}(n, \boldsymbol{C}), \mathfrak{so}(n, \boldsymbol{C}))$ or $(\mathfrak{sl}(2n, \boldsymbol{C}), \mathfrak{sp}(n, \boldsymbol{C}))$ for some integer n. Then the condition* (1) *in Prop.* 1 *holds for any element X of $\mathfrak{q}_0$. In particular every H-orbit of $\mathfrak{q}_0$ has an H-invariant measure.*

In the rest of this section, we consider such a pair $(\mathfrak{g}_0, \mathfrak{h}_0)$ that $(\mathfrak{g}, \mathfrak{h}) =(\mathfrak{sl}(m+n, \boldsymbol{C}), \mathfrak{sl}(m, \boldsymbol{C})+\mathfrak{sl}(n, \boldsymbol{C})+\boldsymbol{C})$ for some integers m, n with $m \geqq n \geqq 1$. To treat this case, we need some preparations. Let X be a nilpotent matrix of size n. If the Jordan's normal form of X is $\begin{bmatrix} J_{p_1} & & \\ & \ddots & \\ & & J_{p_k} \end{bmatrix}$ with $p_1 \geqq p_2 \geqq \cdots \geqq p_k \geqq 1$, $p_1+\cdots+p_k=n$. Here $J_p=\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \cdot & 1 \\ & & & & 0 \end{bmatrix}$ is a $p \times p$ matrix. Then we define a partition $\eta(X)=(p_1, \cdots, p_k)$ of n associated with X. First we take a nilpotent element X of $\mathfrak{q}_0$.

Lemma 3. *Let X be a nilpotent element of $\mathfrak{q}_0$. Then $\mathfrak{h}(X)$ satisfies the condition* (1) *in Prop.* 1 *if and only if the partition $\eta(X)=(p_1, \cdots, p_k)$ satisfies the condition* (2):

(2) *there is a number i such that p_j is odd* (resp. even) *if $j \leqq i$* (resp. $j>i$).

Next we treat an arbitrary element X of $\mathfrak{q}_0$. We consider the Jordan decomposition $X=A+N$, where A is semisimple and N is nilpotent. Then it follows from [5] that A and N are contained in $\mathfrak{q}_0$. Let $\mathfrak{z}$ be the derived algebra of the centralizer of A in $\mathfrak{g}$. Extending σ to $\mathfrak{g}$ as a complex linear involution, we find that $\sigma(\mathfrak{z})=\mathfrak{z}$. Noting that $\mathfrak{z}$ is semisimple, we can conclude that there are simple Lie algebras $\mathfrak{z}_0, \mathfrak{z}_1, \cdots, \mathfrak{z}_{2p}$ of type A such that $\mathfrak{z}=\mathfrak{z}_0+\mathfrak{z}_1+\cdots+\mathfrak{z}_{2p}$ is the direct sum decomposition of $\mathfrak{z}$ into simple factors and that $\sigma(\mathfrak{z}_0)=\mathfrak{z}_0$, $\sigma(\mathfrak{z}_{2i-1})=\mathfrak{z}_{2i}$ $(1 \leqq i \leqq p)$. Then it follows that $(\mathfrak{z}_0, \mathfrak{z}_0 \cap \mathfrak{h})=(\mathfrak{sl}(m_1+n_1, \boldsymbol{C}), \mathfrak{sl}(m_1, \boldsymbol{C}) +\mathfrak{sl}(n_1, \boldsymbol{C})+\boldsymbol{C})$ for some integers m_1, n_1 $(m_1 \geqq n_1 \geqq 1)$. We decompose $N=N_0+N_1+\cdots+N_p$ such that N_0 is in $\mathfrak{z}_0 \cap \mathfrak{q}$ and N_i is in $(\mathfrak{z}_{2i-1}+\mathfrak{z}_{2i}) \cap \mathfrak{q}$ $(1 \leqq i \leqq p)$.

Theorem 4. *Let X be an arbitrary element of $\mathfrak{q}_0$ and use the notation introduced above. Then $\mathcal{O}(X)$ has an H-invariant measure if* *and only if* $\eta(N_0)$ *satisfies the condition* (2) *in Lemma* 3.

§3. An extension of the invariant measures. In this section, we assume that $X \in \mathfrak{q}_0$ is nilpotent and that $\mathcal{O}(X)$ has an H-invariant measure $d\mu$. We examine the possibility of extending $d\mu$ to an H-invariant distribution on $\mathfrak{q}_0$.

Let $X_0 \in \mathfrak{q}_0$ be nilpotent and assume that $\mathcal{O}(X_0)$ has an H-invariant measure $d\mu$. We take a normal S-triple (A, X_0, Y_0), that is, (A, X_0, Y_0) is an S-triple and $A \in \mathfrak{h}_0$, X_0, $Y_0 \in \mathfrak{q}_0$ (cf. [5]). Define $\mathfrak{h}_0(i)=\{Z \in \mathfrak{h}_0; [A, Z]=iZ\}$, $\mathfrak{q}_0(i)=\{Z \in \mathfrak{q}_0; [A, Z]=iZ\}$ for any integer i. For later use, we set $\mathfrak{n}_{\mathfrak{h}}=\oplus_{i>0}\mathfrak{h}_0(i)$, $\mathfrak{n}=\oplus_{i>2}\mathfrak{q}_0(i)$. Then $\mathfrak{p}=\mathfrak{h}_0(0)+\mathfrak{n}_{\mathfrak{h}}$ is a parabolic subalgebra of $\mathfrak{h}_0$. Let P denote the parabolic subgroup of H corresponding to $\mathfrak{p}$. If M_{X_0} is the centralizer of A in P and if N_P is the unipotent radical of P, then $P=M_{X_0}N_P$ is a Levi decomposition of P. By definition, $\mathfrak{h}_0(0)$ and $\mathfrak{n}_{\mathfrak{h}}$ are the Lie algebras of M_{X_0} and N_P, respectively. Since P acts on $\mathfrak{n}_{\mathfrak{h}}$ and $\mathfrak{n}$, we can define functions $\delta(p)$, $\gamma_1(p)$ and $\gamma_2(p)$ on P by $\delta(p)=|\det(Ad_H(p)|\mathfrak{n}_{\mathfrak{h}})|$, $\gamma_1(p)=|\det(Ad_G(p)|\mathfrak{q}_0(2))|$ and $\gamma_2(p)=|\det(Ad_G(p)|\mathfrak{n})|$ for any $p \in P$. We note that $\delta(pn)=\delta(p)$ and $\gamma_i(pn)=\gamma_i(p)$ $(i=1, 2)$ for any $p \in P$ and $n \in N_P$. By an argument similar to the proof of [4, Lemma 1], we show that $N_P\cdot X_0=X_0+\mathfrak{n}$ and $P\cdot X_0=V+\mathfrak{n}$, where $V=M_{X_0}\cdot X_0$ is an open subset of $\mathfrak{q}_0(2)$. Then the following lemma is proved by direct calculation.

Lemma 5. *There are homogeneous polynomials* $f_1(X)$, $f_2(X)$ *on* $\mathfrak{q}_0(2)$ *and positive rational numbers* r_1, r_2 *with the condition* (3) *below*:

(3) *Define a function* $\Phi(X)$ *by* $\Phi(X)=|f_1(X)|^{r_1}/|f_2(X)|^{r_2}$ *on* V. *Then* $\Phi(m\cdot X)=(\delta(m)/\gamma_1(m)\gamma_2(m))\Phi(X)$ *for any* $m \in M_{X_0}$, $X \in V$.

Using Lemma 5, we examine whether the H-invariant measure $d\mu$ on $\mathcal{O}(X_0)$ is rewritten as an orbital integral on $\mathfrak{q}_0$ or not. For this purpose, we take a Cartan involution θ of $\mathfrak{g}_0$ commuting with σ (cf. [1]) and denote by K the maximal compact subgroup of G corresponding to θ. Then $K_H=K\cap H$ is a maximal compact subgroup of H.

Proposition 6. *If* $F \in C_0^\infty(\mathfrak{q}_0)$ *satisfies the condition that*

$$(\mathrm{Supp}\, F)\cap \mathcal{O}(X_0)$$

is compact, then $\int_{\mathcal{O}(X_0)} F d\mu$ *is convergent and the following equality* (4) *holds*:

$$(4) \quad \int_{\mathcal{O}(X_0)} F d\mu = \int_{K_H} dk \int_{V+\mathfrak{n}} F(k(X+Z))\Phi(X)dXdZ.$$

Here dk *is a Haar measure on* K_H *and* dX, dZ *are Euclidean measures on* $\mathfrak{q}_0(2)$, $\mathfrak{n}$, *respectively.*

If $f_2(X)$ is a non-zero constant, it easily follows from Prop. 6 that the right-hand side of the equation (4) is convergent for any $F \in C_0^\infty(\mathfrak{q}_0)$ and therefore defines a Radon measure on $\mathfrak{q}_0$. But this condition on $f_2(X)$ does not hold in general. Hence in order to extend the measure

$d\mu$ to $\mathfrak{q}_0$ as an H-invariant distribution, we must regularize the function $\Phi(X)$. Let S be a distribution on $\mathfrak{q}_0(2)+\mathfrak{n}$ such that for any $f\in C_0^\infty(\mathfrak{q}_0(2)+\mathfrak{n})$, we have $S(f^m)=\delta(m)S(f)$ $(\forall m\in M_{X_0})$, where $f^m(Z)=f(m^{-1}\cdot Z)$. Such a distribution S always exists by regularizing the function $\Phi(X)$ defined on V (cf. [3]). We note that S is *not* uniquely determined by $\Phi(X)$ in general.

Theorem 7. *With the distribution S on $\mathfrak{q}_0(2)+\mathfrak{n}$ defined above, we can associate a distribution T on $\mathfrak{q}_0$ by $T(f)=S(\bar{f})$ for any $f\in C_0^\infty(\mathfrak{q}_0)$. Here $\bar{f}$ is the function on $\mathfrak{q}_0(2)+\mathfrak{n}$ defined by $\bar{f}(X+Z)=\int_{K_H} f(k(X+Z))dk$. Moreover T is H-invariant and for any H-invarint polynomial P on $\mathfrak{q}_0$, we have $PT=P(0)T$.*

The extended version of the results of this note including their proofs will be published elsewhere.

References

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