27. On the Structure of Cohomology Groups attached to the Integral of Certain Many-Valued Analytic Functions

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0. Introduction. The present note is a brief summary of our forthcoming paper [7].

Let P_j $(1 \leq j \leq m)$ be non-zero polynomials in n complex variables $z=(z_1, \cdots, z_n)$ and A_j $(1 \leq j \leq m)$ be linear mappings of a finite dimensional complex vector space V. We consider the connection form $\omega = \sum_{j=1}^{m} A_j(dP_j/P_j)$ *which satisfies the integrability condition* $\omega \wedge \omega = 0$. Let D_j be the divisor of $\mathbf{C}^n$ defined by P_j for $1 \leq j \leq m$ and D be the divisor defined by the product $P = P_1 \cdots P_m$. We denote by $\Omega^p_{X^{an}}$ the sheaf of germs of holomorphic p-forms on the complex manifold $X = \mathbf{C}^n - D$. Then the 1-form ω determines an integrable connection ∇_ω on $\Omega^{\cdot}_{X^{an}} \otimes V$ as follows:

$$\nabla_\omega \varphi := d\varphi + \omega \wedge \varphi$$

for each local section φ of $\Omega^p_{X^{an}} \otimes V$. We denote by $\mathcal{S}_\omega$ the complex local system on X defined by the local horizontal sections of ∇_ω. Let $\Omega^p(*D)$ be the set of *rational* p-forms which are holomorphic on X; then we denote by $(\Omega^{\cdot}(*D) \otimes V, \nabla_\omega)$ the complex

$$0 \longrightarrow \Omega^0(*D) \otimes V \xrightarrow{\nabla_\omega} \Omega^1(*D) \otimes V \xrightarrow{\nabla_\omega} \cdots \xrightarrow{\nabla_\omega} \Omega^n(*D) \otimes V \longrightarrow 0.$$

Since X is affine, by the comparison theorem of Grothendieck and Deligne we have isomorphisms

$$H^p(X, \mathcal{S}_\omega) \xrightarrow{\sim} H^p(\Omega^{\cdot}(*D) \otimes V, \nabla_\omega) \quad \text{for} \quad 0 \leq p \leq n.$$

After K. Aomoto, we call the complex $(\Omega^{\cdot}(*D) \otimes V, \nabla_\omega)$ the *twisted rational de Rham complex.*

In the present note, we discuss the vanishing theorems for the twisted rational de Rham cohomology groups $H^p(\Omega^{\cdot}(*D) \otimes V, \nabla_\omega)$ under certain algebraic conditions on the divisors D_j $(1 \leq j \leq m)$ and on the residue matrices A_j $(1 \leq j \leq m)$. This type of studies of cohomology groups of $\mathbf{C}^n - D$ with coefficients in local systems has been made by K. Aomoto from the viewpoint of differential equations ([1]–[4]) and by A. Hattori and T. Kimura from the topological point of view ([5] and [6]). We extend the results of the papers cited above to complex

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local systems $\mathcal{S}_\omega$ of any rank and to larger classes of divisors than those defined by linear polynomials. As an application of our method, we can give a complete description of the cohomology group $H^n(\Omega^\cdot(*D)\otimes V, \nabla_\omega)$ in the case where D_j $(1\le j\le m,\ m\ge n+1)$ are hyperplanes in general position, which gives a *positive answer to the conjecture proposed by* K. Aomoto [1].

1. After K. Saito [8], we say that a rational p-form ψ in $\Omega^p(*D)$ is *generically logarithmic along* D if $P\psi$ and $Pd\psi$ are polynomial forms. We denote by $\Omega^p(\log D)$ the set of rational p-forms generically logarithmic along D. Let $\boldsymbol{D}=\{D_j \mid 1\le j\le m\}$; then a rational p-form ψ in $\Omega^p(*D)$ is said to be *logarithmic with respect to* $\boldsymbol{D}$ if it can be written in the form

$$\psi=\sum_{\nu=0}^{N}\sum_{1\le j_1<\cdots<j_\nu\le m}\frac{dP_{j_1}}{P_{j_1}}\wedge\cdots\wedge\frac{dP_{j_\nu}}{P_{j_\nu}}\wedge\psi_{j_1\cdots j_\nu}$$

where $N=\min(m,p)$ and each $\psi_{j_1\cdots j_\nu}$ is a polynomial $(p-\nu)$-form for $1\le j_1<\cdots<j_\nu\le m$. We denote by $\Omega^p\langle \boldsymbol{D}\rangle$ the set of rational p-forms logarithmic with respect to $\boldsymbol{D}$. We remark that $\Omega^p\langle \boldsymbol{D}\rangle$ is contained in $\Omega^p(\log D)$ for $0\le p\le n$.

Let $\mathbf{C}[z]$ be the polynomial ring of n variables $z=(z_1,\cdots,z_n)$ over $\mathbf{C}$. For a sequence $Q_1,\cdots,Q_r$ of polynomials in $\mathbf{C}[z]$, we denote by $(dQ_1\wedge\cdots\wedge dQ_r, Q_1,\cdots,Q_r)$ the ideal of $\mathbf{C}[z]$ generated by $Q_1,\cdots,Q_r$ and the minors of size r of the Jacobian matrix $(\partial Q_k/\partial z_i)$ $(1\le k\le r,\ 1\le i\le n)$. Recall that a sequence $Q_1,\cdots,Q_s$ of elements of $\mathbf{C}[z]$ is said to be *regular* if each Q_k $(1\le k\le s)$ satisfies the following condition: If F is an element of $\mathbf{C}[z]$ such that Q_kF is in the ideal $(Q_1,\cdots,Q_{k-1})$, then F itself belongs to $(Q_1,\cdots,Q_{k-1})$.

Let q be an integer with $1\le q\le n+1$. Then we say that a set $\{P_j \mid 1\le j\le m\}$ of polynomials in $\mathbf{C}[z]$ *satisfies the condition* $\boldsymbol{C}(q)$ if the following holds: (1) *The height of the ideal* $(dQ_1\wedge\cdots\wedge dQ_r, Q_1,\cdots,Q_r)$ *is not less than* q *for any* r *polynomials* $Q_1,\cdots,Q_r$ *of* $\{P_j \mid 1\le j\le m\}$ *where* $1\le r\le\min(m,q-1)$. (2) $Q_1,\cdots,Q_s$ *form a regular sequence for any* s *polynomials* $Q_1,\cdots,Q_s$ *of* $\{P_j \mid 1\le j\le m\}$ *where* $1\le s\le\min(m,q)$. For convenience, we define the height of the ideal $\mathbf{C}[z]$ to be $n+1$.

We remark that both $\Omega^\cdot\langle \boldsymbol{D}\rangle$ and $\Omega^\cdot(\log D)$ are closed under the exterior differentiation and the exterior product with the 1-form dP_j/P_j for $1\le j\le m$; hence we have two logarithmic subcomplexes $(\Omega^\cdot\langle \boldsymbol{D}\rangle\otimes V, \nabla_\omega)$ and $(\Omega^\cdot(\log D)\otimes V, \nabla_\omega)$ of the twisted rational de Rham complex $(\Omega^\cdot(*D)\otimes V, \nabla_\omega)$. Then we have

Theorem 1. *Suppose that the eigenvalues of the residue matrices* A_j $(1\le j\le m)$ *are different from positive integers.*

(i) *If the set* $\{P_j \mid 1\le j\le m\}$ *of polynomials satisfies the condition* $C(q)$ *for an integer* q *with* $1\le q\le n$, *then we have natural isomorphisms*

$$H^p(\Omega^{\cdot}\langle D\rangle\otimes V, \nabla_\omega)\xrightarrow{\sim}H^p(\Omega^{\cdot}(*D)\otimes V, \nabla_\omega) \quad for \quad 0\le p\le q-2.$$

(ii) *If the set* $\{P_j\,|\,1\le j\le m\}$ *of polynomials satisfies the condition* $C(n+1)$, *then we have natural isomorphisms*

$$H^p(\Omega^{\cdot}\langle D\rangle\otimes V, \nabla_\omega)\xrightarrow{\sim}H^p(\Omega^{\cdot}(*D)\otimes V, \nabla_\omega) \quad for \quad 0\le p\le n.$$

2. We fix an n-tuple $\rho=(\rho_1,\cdots,\rho_n)\ne(0,\cdots,0)$ of non-negative integers. Then the ρ-degree of a monomial $z_1^{\alpha_1}\cdots z_n^{\alpha_n}$ is defined as the sum $\mu=\sum_{i=1}^n \rho_i\alpha_i$. A linear combination of monomials of ρ-degree μ over $\mathbf{C}$ is called a weighted homogeneous polynomial of weight ρ of ρ-degree μ.

Let P_j $(1\le j\le m)$ be non-zero polynomials in $\mathbf{C}[z]$ which are linear combinations of monomials of ρ-degree at most l_j. We denote by $\bar{P}_j$ the weighted homogeneous part of P_j of ρ-degree l_j and suppose that each $\bar{P}_j$ is not zero for $1\le j\le m$. Then we have

Theorem 2. *Let q be an integer with $1\le q\le n$ and suppose that the set $\{\bar{P}_j\,|\,1\le j\le m\}$ of weighted homogeneous polynomials satisfies the condition $C(q)$. Then the following holds:*

(i) *Let $l=\sum_{j=1}^m l_j$. If the eigenvalues of the linear mapping $\sum_{j=1}^m l_jA_j$ of V are different from the integers $l, l-1, l-2, \cdots$, then the cohomology groups $H^p(\Omega^{\cdot}(\log D)\otimes V, \nabla_\omega)$ vanish for $0\le p<q$.*

(ii) *If the eigenvalues of the linear mapping $\sum_{j=1}^m l_jA_j$ of V are different from non-positive integers $0, -1, -2, \cdots$, then the cohomology groups $H^p(\Omega^{\cdot}\langle D\rangle\otimes V, \nabla_\omega)$ vanish for $0\le p<q$.*

3. Remarking that $\Omega^{\cdot}(*D)\otimes V$ is the union of $P^{-k}\Omega^{\cdot}(\log D)\otimes V$ $(k=1,2,\cdots)$, we have

$$H^p(\Omega^{\cdot}(*D)\otimes V, \nabla_\omega)=\text{ind.}\lim H^p(P^{-k}\Omega^{\cdot}(\log D)\otimes V, \nabla_\omega)$$

for $0\le p\le n$. On the other hand, the subcomplex $(P^{-k}\Omega^{\cdot}(\log D)\otimes V, \nabla_\omega)$ is isomorphic to the complex $(\Omega^{\cdot}(\log D)\otimes V, \nabla_{\omega(k)})$ where $\omega(k)$ is the connection form $\sum_{j=1}^m (A_j-k\cdot id_V)(dP_j/P_j)$. Then combining Theorem 2 with the above remarks, we obtain

Main theorem. *Let q be an integer with $1\le q\le n$. Suppose that the set $\{\bar{P}_j\,|\,1\le j\le m\}$ of weighted homogeneous polynomials satisfies the condition $C(q)$. If the eigenvalues of the linear mapping $\sum_{j=1}^m l_jA_j$ of V are different from rational integers, then the twisted rational de Rham cohomology groups $H^p(X, \mathcal{S}_\omega)=H^p(\Omega^{\cdot}(*D)\otimes V, \nabla_\omega)$ vanish for $0\le p<q$.*

In the case where D_j $(1\le j\le m, m\ge n+1)$ are hyperplanes in general position, we can determine the n-th cohomology group.

Theorem 3. *Let $m\ge n+1$ and let P_j $(1\le j\le m)$ be linear polynomials such that the hyperplanes D_j $(1\le j\le m)$ are in general position. If the eigenvalues of the linear mapping $\sum_{j=1}^m A_j$ of V are different from non-positive integers and those of each A_j $(1\le j\le m)$ different from positive integers, then we have*

$$H^p(X, \mathcal{S}_\omega)=H^p(\Omega^\cdot(*D)\otimes V, \nabla_\omega)=0 \quad for \quad 0\leq p\leq n-1.$$

Moreover, the n-th cohomology group $H^n(X, \mathcal{S}_\omega)=H^n(\Omega^\cdot(*D)\otimes V, \nabla_\omega)$ *is given by*

$$G_0(\Omega^n\langle \boldsymbol{D}\rangle\otimes V)/\omega\wedge G_0(\Omega^{n-1}\langle \boldsymbol{D}\rangle\otimes V)$$

where $G_0(\Omega^p\langle \boldsymbol{D}\rangle\otimes V)$ *is the complex vector space which consists of all linear combinations of* $dP_{j_1}/P_{j_1}\wedge\cdots\wedge dP_{j_p}/P_{j_p}$ $(1\leq j_1<\cdots<j_p\leq m)$ *with coefficients in* V.

Theorem 3 gives a positive answer to the conjecture proposed by K. Aomoto [1].

References

[1] K. Aomoto: Un théorème du type Matsushima-Murakami concernant l'intégrale des fonction multiformes. J. Math. Pures Appl., **52**, 1–11 (1973).

[1] ——: Vanishing of cohomology attached to certain many valued meromorphic functions. J. Math. Soc. Japan, **27**, 248–255 (1975).

[3] ——: Les équations aux différences linéaires et les intégrales des fonctions multiformes. J. Fac. Sci. Univ. Tokyo, Sec. IA, **22**, 271–297 (1975); Une correction et un complément à l'article "Les équations aux différences . . .". ibid., **26**, 519–523 (1979).

[4] ——: Les intégrales des fonctions multiformes (I) (unpublished).

[5] A. Hattori and T. Kimura: On the Euler integral representations of hypergeometric functions in several variables. J. Math. Soc. Japan, **26**, 1–16 (1974).

[6] A. Hattori: Topology of $\mathbf{C}^n$ minus a finite number of affine hyperplanes in general position. J. Fac. Sci. Univ. Tokyo, Sec. IA, **22**, 205–219 (1975).

[7] M. Kita and M. Noumi: On the structure of cohomology groups attached to the integral of certain many-valued analytic functions (to appear).

[8] K. Saito: Theory of logarithmic differential forms and logarithmic vector fields. J. Fac. Sci. Univ. Tokyo, Sec. IA, **27**, 285–291 (1980).