25. On Fourier Coefficients and Certain "Periods" of Modular Forms

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In this paper, we shall show a relation between Fourier coefficients of modular forms of half integral weight and certain "periods" of modular forms of integral weight.

We denote the upper half plane by H. Put $e(z)=\exp(2\pi iz)$, $\theta(z)=\sum_{n=-\infty}^{\infty} e(n^2z)$ and $k=2m+1$ for an integer m. For $M=\begin{pmatrix} ab \\ cd \end{pmatrix} \in GL(2, \boldsymbol{R})$ and a function $f(z)$ on H we put $Mz=(az+b)/(cz+d)$, $j(M,z)=\theta(Mz)/\theta(z)$ and $f(z)|_z[M]_{k/2}=j(M,z)^{-k}f(Mz)$. We denote the space of cusp forms of weight $2m$ (resp. k/2) for $\Gamma_0(2)$ (resp. $\Gamma_0(4)$) by U (resp. V). For a rational number r and a discrete subgroup Γ, put

$$\langle f(z), g(z)\,;\, \Gamma, r, z\rangle=\int_{\Gamma\backslash H} f(z)\overline{g(z)}y^{r-2}dxdy,\ (z=x+iy).$$

We define an operation of $g \in G=SL(2,\boldsymbol{R})$ on $\boldsymbol{R}^3$ by $g(x,y,z)=(x',y',z')$ where $\begin{pmatrix} x' & y'/2 \\ y'/2 & z' \end{pmatrix}=g\begin{pmatrix} x & y/2 \\ y/2 & z \end{pmatrix}^t g$. Put $\chi_t(n)=\left(\frac{-1}{n}\right)^m\left(\frac{t}{n}\right)$ with Shimura's symbol $(-)$ in [2]. For $(a,b,c)\in \boldsymbol{R}^3$, put $h(a,b,c)=(a-ib-c)^m e(i(2a^2+b^2+2c^2)/p)$. Let p be an odd prime and let $L=(4p)^{-1}\boldsymbol{Z}\times\boldsymbol{Z}\times p\boldsymbol{Z}$. We assume $p\equiv(-1)^{m+1} \bmod 4$ throughout this paper. For $z=x+iy$ and $g\in G$, define

$$\theta_1(z,g)=\sum_{(a,b,c)\in L}\chi_p(4pa)y^{(3-k)/4}h(\sqrt{py}\ g^{-1}(a,b,c))e(x(b^2-4ac)).$$

For $w=u+iv$, put $g_w=\begin{pmatrix} \sqrt{v} & u/\sqrt{v} \\ 0 & 1/\sqrt{v} \end{pmatrix}$ and let $\theta_2(z,w)=v^{-m}\theta_1(z,g_w)$. Let $\Gamma_0(n,p)=\left\{\begin{pmatrix} ab \\ cd \end{pmatrix}\in\Gamma_0(n)\,|\, b\equiv 0 \bmod p\right\}$ and let $\Gamma_0(4,p)\backslash\Gamma_0(4)=\{M_1, M_2, \cdots, M_{p+1}\}$. Put $\theta_3(z,w)=\sum_{i=1}^{p+1} p^{-m}\theta_2(z/p, w/p)|_z[M_i]_{k/2}$. The equality in [6, p. 154, line 7] implies that $\theta_3(z,w)$ satisfies transformation formulas for $\Gamma_0(2,p)$ as a function of w. Moreover, using [3, Prop. 1.6] we can express θ_3 as a linear combination of some θ series. Though the present case is more complicated, we get in the same way as in the first paragraph in [6, p. 154] the following proposition.

Proposition 1. $\theta_3(z,w+1)=\theta_3(z,w)$.

For $f(z)=\sum_{n=1}^{\infty} a(n)e(nz)\in V$, put $F\{f\}_t(z)=\sum_{n=1}^{\infty} A_t(n)e(nz)$ with $A_t(n)$ determined by the relation

$$\sum_{n=1}^{\infty} A_t(n)n^{-s}=\left(\sum_{n=1}^{\infty} a(tn^2)n^{-s}\right)\left(\sum_{n=1}^{\infty}\chi_t(n)n^{m-1-s}\right).$$

Then, by [6] we get

Proposition 2. *For* $f \in V$, $\langle f(z), \theta_3(z,w); \Gamma_0(4), k/2, z\rangle = dF\{f\}_p(w)$ *where* $d = p^{m/2+3/2}2^{-m+3}(-i)^m\overline{\mathfrak{G}(\chi_p)}^{-1}$ *with the Gaussian sum* $\mathfrak{G}(\chi_p)$ *of* χ_p.

Let $\{h_i(z)\}$ be orthonormal basis of V and put $K(z,w) = \sum_i h_i(z)\overline{h_i(w)}$. Then, K is the reproducing kernel function of V. Put $L(z,w) = \langle K(z,z'), \overline{\theta_3(z',w)}; \Gamma_0(4), k/2, z'\rangle$. Then, it follows from Proposition 2 that

(1) $$\langle f(z), L(z,w); \Gamma_0(4), k/2, z\rangle = dF\{f\}_p(w).$$

By [1] and [5], we get $V = (\oplus_i \tilde{O}_i) \oplus (\oplus_j \tilde{N}_j)$; $U = (\oplus_i O_i) \oplus (\oplus_j N_j)$; $\dim \tilde{O}_i = \dim O_i = 2$; $\dim \tilde{N}_j = \dim N_j = 1$; $\tilde{O}_i \sim O_i$, $\tilde{N}_j \sim N_j$ as Hecke algebra modules. Let G_j be the element in N_j whose first Fourier coefficient is unity. Then, G_j is a newform. Let $a_j(l)$ be the l-th Fourier coefficient of a base g_j in $\tilde{N}_j$. Then, $F\{g_j\}_p = a_j(p)G_j$. Denote $\langle\ ,\ ; \Gamma_0(4), k/2, z\rangle$ simply by $\langle\ ,\ \rangle$. Let $f_{i,1}$, $f_{i,2}$ be an orthonormal basis of $\tilde{O}_i$. Then, by [5] and (1) we get

Proposition 3.

$$L(z,w) = \bar{d} \sum_j g_j(z)\overline{a_j(p)G_j(w)}\langle g_j, g_j\rangle^{-1} + \bar{d} \sum_{i,n} f_{i,n}(z)\overline{F\{f_{i,n}\}_p(w)}.$$

It is rather easy to see that

$$\begin{aligned}
&\langle L(z,w), \overline{G_j(w)}; \Gamma_0(2), 2m, w\rangle \\
&\quad = \langle K(z,z'), \langle \overline{\theta_3(z',w)}, G_j(w); \Gamma_0(2), 2m, w\rangle; \Gamma_0(4), k/2, z'\rangle \\
&\quad = \langle \theta_3(z,w), \overline{G_j(w)}; \Gamma_0(2), 2m, w\rangle.
\end{aligned}$$

Thus, we get

Proposition 4.

$$\begin{aligned}
&\langle \theta_3(z,w), \overline{G_j(w)}; \Gamma_0(2), 2m, w\rangle\langle g_j, g_j\rangle \\
&\quad = \overline{d a_j(p)} g_j(z)\langle G_j(w), G_j(w); \Gamma_0(2), 2m, w\rangle.
\end{aligned}$$

By [3] we can calculate Fourier coefficients of the function (of z) in the left side of the above equality. For a positive integer n, put

$S'_n = \{(a,b,c) \in (1/4)\boldsymbol{Z}\times\boldsymbol{Z}\times\boldsymbol{Z} \mid (4a, 4p) = 1,\ b^2 - 4ac = np\}$,

$S''_n = \{(a,b,c) \in (p/4)\boldsymbol{Z}\times p\boldsymbol{Z}\times\boldsymbol{Z} \mid (4a, 4) = 1,\ (c,p) = 1,\ b^2 - 4ac = np\}$,

and $S_n = S'_n \cup S''_n$. Denote by ψ_r the primitive real character modulo r. For $t = (a,b,c) \in S_n$, put $\xi(t) = \psi_{4p}(4a)$ when $t \in S'_n$, and $\xi(t) = \psi_p(c)\psi_4(4a)$ when $t \in S''_n$, denote by T_n the complete set of representatives of $\Gamma_0(2)$-equivalence classes in S_n, and let $C(t, \Gamma_0(2))$ be the geodesic or the rectifiable curve on H defined in p. 101 of [3] for a binary quadratic form $ax^2 + bxy + cy^2$. Denote also $\langle\ ,\ ; \Gamma_0(2), 2m, w\rangle$ simply by $\langle\ ,\ \rangle$. Then, by comparing Fourier coefficients of both sides of the equality in Proposition 4, we get

Theorem 1.

$$\begin{aligned}
&2^{-m+3}(-i)^m\sqrt{p}\,\overline{\mathfrak{G}(\chi_p)}^{-1}\overline{a_j(p)}a_j(n)\langle G_j, G_j\rangle\langle g_j, g_j\rangle^{-1} \\
&\quad = \sum_{t=(a,b,c)\in T_n} \xi(t) \int_{C(t,\Gamma_0(2))} G_j(z)(a - bz + cz^2)^{m-1}dz.
\end{aligned}$$

For $G(z) = \sum_{n=1}^{\infty} c(n)e(nz)$ and a character χ, put $L(s, G, \chi) = \sum_{n=1}^{\infty} c(n)\chi(n)n^{-s}$. It is easy to see that $T_p = \{t(k) = (k/4, p, 0) \mid k \bmod 4p,$

$(k, 4p)=1\}$. Since $C(t(k), \Gamma_0(2))$ is the geodesic line from $i\infty$ to $k/4p$, the summand of the equality in Theorem 1 becomes

$$\chi_p(k)\int_{i\infty}^{k/4p} G_j(z)(-k/4p+z)^{m-1}dz$$

in case $n=p$, and we especially get the following theorem due to Waldspurger:

Theorem 2.

$$2\langle G_j, G_j\rangle\langle g_j, g_j\rangle^{-1}|a_j(p)|^2=(m-1)!\,\pi^{-m}p^{m-1/2}L(m, G_j, \chi_p).$$

We note that Waldspurger proved Theorem 2 under more general settings and that our method also can apply to the general cases.

Finally we correct errata related to our present results in the previous papers. The constant c in Theorem of [6] is not correct. The correct value is $(-1)^{\lambda}N^{\lambda/2+1/4}2^{-3\lambda+2}$. $\left(\frac{s^2-2n}{p}\right)^{\nu_p}$ on the eighth line in p. 187 of [5] should read $\left(\frac{2s^2-4n}{p}\right)^{\nu_p}$.

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