22. *A Characterization of the Intersection Form of a Milnor's Fiber for a Function with an Isolated Critical Point*

By Kyoji SAITO

Research Institute for Mathematical Sciences, Kyoto University

(Communicated by Kunihiko KODAIRA, M. J. A., Feb. 12, 1982)

§ 1. Introduction and the statements of the main results. Let $f: \boldsymbol{C}^{n+1}, 0 \to \boldsymbol{C}, 0$ be a germ of a holomorphic function at $0 \in \boldsymbol{C}^{n+1}$ with an isolated critical point. Due to Milnor [2], for r and ε sufficiently small with $0<\varepsilon \ll r \ll 1$, the restriction

$$f: \{x \in \boldsymbol{C}^{n+1} : |x|<r\} \cap \{|f|=\varepsilon\} \longrightarrow \{t \in \boldsymbol{C} : |t|=\varepsilon\}$$

of f defines a fibration whose general fiber F is a bouquet of n-spheres so that the middle homology group $H_n(F, \boldsymbol{Z})$ is nonvanishing.

Using Poincaré duality $H_n(F, \boldsymbol{Z}) \simeq H^n(F, \partial F, \boldsymbol{Z})$, one gets an intersection form $\langle\ ,\ \rangle: H_n(F, \boldsymbol{Z}) \times H_n(F, \boldsymbol{Z}) \to \boldsymbol{Z}$, which is symmetric or skew-symmetric according as n is even or odd.

For a computation of the intersection form, we used in [3] the following fact.

Theorem 1. *A complex valued bilinear form B on $H_n(F, \boldsymbol{Z}) \otimes \boldsymbol{C}$ is a constant multiple of the intersection form if B is invariant under the total monodromy group action on $H_n(F, \boldsymbol{Z})$, except for the case when f at 0 is nondegenerate* (i.e. ordinary double point) *and n is odd. Here the total monodromy group is by definition the image of the fundamental group of the complement of the discriminant loci of a universal unfolding of f.*

Since this fact seems still not generally well-known, we publish it here with a proof separately from [3]. In § 2 we give a somewhat abstract lemma characterizing invariant bilinear forms.

§ 2. The uniqueness lemma for an invariant bilinear form. Let V be a vector space over a field k with ch $k \neq 2$ and let $\langle\ ,\ \rangle: V \times V \to k$ be a k-bilinear form which is either symmetric or skew-symmetric.

Let A be a subset of V. In case $\langle\ ,\ \rangle$ is symmetric, we assume $\langle e, e\rangle = 2$ for all $e \in A$. Let us associate the graph $\Gamma(A)$ to such A as follows. The set of vertices of $\Gamma(A)$ is in a one-to-one correspondence to A so that we identify them. Two vertices e and e' of A are connected by a 1-simplex if and only if $\langle e, e'\rangle \neq 0$.

Let $W(A)$ be the subgroup of $GL(V)$ generated by the set of reflexions σ_e for $e \in A$, where

$$\sigma_e(u) := u - \langle u, e\rangle e \qquad \text{for } u \in V.$$

One checks easily that the group $W(A)$ leaves the form $\langle\ ,\ \rangle$ invariant. Now we formulate our lemma.

Lemma 2. *Assume that* i) *the graph* $\Gamma(A)$ *is connected and that* ii) V *is generated by the elements of* A *over* k. *Then a bilinear form* $B: V\times V\to k$ *is a constant multiple of* $\langle\ ,\ \rangle$, *if it is invariant under the action of* W, i.e.

$$B(u,v)=B(wu,wv) \qquad \text{for } \forall u, v\in V,\ \forall w\in W,$$

except for the case when $\sharp A=1$ *and* $\langle\ ,\ \rangle$ *is skew-symmetric.*

Proof. For $e\in A$, the relation $B(u,v)=B(\sigma_e u,\sigma_e v)$ implies the relation

1) $\langle u,e\rangle B(e,v)+\langle v,e\rangle B(u,e)-\langle u,e\rangle\langle v,e\rangle B(e,e)=0$ for all $u, v\in V$.

The assumptions on A in the lemma imply the existence of $e'\in A$ such that $\langle e',e\rangle\neq 0$. By taking v in 1) to be e' we get the formula

2) $B(u,e)=\langle e',e\rangle^{-1}\{\langle e',e\rangle B(e,e)-B(e,e')\}\langle u,e\rangle$.

In other words, for any $e\in A$, there exists a constant $\alpha(e)\in k$ such that

2)′ $B(u,e)=\alpha(e)\langle u,e\rangle$ for all $u\in V$.

An analogous computation shows also that for any $e\in A$, there exists a constant $\beta(e)\in k$ such that

3) $B(e,v)=\beta(e)\langle e,v\rangle$ for all $v\in V$.

Let us check that $\alpha(e)=\beta(e)$ for any $e\in A$.
If $\langle\ ,\ \rangle$ is symmetric, it follows from the facts $B(e,e)=\alpha(e)\langle e,e\rangle =\beta(e)\langle e,e\rangle$ and $\langle e,e\rangle=2$. If $\langle\ ,\ \rangle$ is skew-symmetric $B(e,e)=\alpha(e)\langle e,e\rangle =0$. Then substituting 2)′ and 3) in 1), we obtain $\langle u,e\rangle\langle v,e\rangle(-\beta(e)+\alpha(e))=0$ $\forall u, v$, which implies $\alpha(e)=\beta(e)$.

For e and $e'\in A$ let us compute $B(e',e)=\alpha(e)\langle e',e\rangle=\beta(e')\langle e',e\rangle$. If e and e' are combined in $\Gamma(A)$ i.e. $\langle e',e\rangle\neq 0$ then one gets

$$\alpha(e)=\beta(e').$$

Since $\Gamma(A)$ is connected (assumption i)), $\alpha(e)=\beta(e)$ is a constant $\gamma\in k$ independent of $e\in A$. The second assumption that A generates V then implies that

$$B(u,v)=\gamma\langle u,v\rangle \qquad \text{for all } u, v\in V. \qquad \text{Q.E.D.}$$

§ 3. A proof of Theorem 1. In Lemma 2, take V to be $H_n(F,\boldsymbol{Z})\otimes\boldsymbol{C}$ and take $(-1)^{n(n-1)/2}\langle\ ,\ \rangle$ to be the intersection form.

Let $A=\{e_1,\cdots,e_\mu\}$ be a strongly distinguished basis of $H_n(F,\boldsymbol{Z})$ which automatically satisfies the condition ii) of Lemma 2 (cf. Appendix of Brieskorn [1]). The condition i) of Lemma 2 is also automatically satisfied, since the discriminant of a universal unfolding of f is irreducible, and its generic singularity is a cusp of (2, 3) type. The Picard-Lefschetz formula says that $W(A)$ is the total monodromy group.

The exceptional case in Lemma 2 corresponds to the exceptional case in Theorem 1.

The author is grateful to M. Saito who asked to publish this result.

References

[1] Brieskorn, E.: Die Monodromie der isolierten Singularitäten von Hyperflächen. Manuscripta math., **2**, 103–160 (1970).

[2] Milnor, J.: Singular points of complex hypersurfaces. Ann. of Math. Studies, No. 61, Princeton (1968).

[3] Saito, K.: On periods of primitive integrals. Harvard (1980) (preprint).