21. C_l-Metrics on Spheres

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1. Let (M, g) be a riemannian manifold. Then we call g a C_l-metric if all of its geodesics are closed and have the common length l. As is well-known, the standard metric on the unit sphere S^n is a $C_{2\pi}$-metric. Suppose $\{g_t\}$ is a one-parameter family of $C_{2\pi}$-metrics on S^n such that g_0 is the standard one. Put

$$\frac{d}{dt} g_t|_{t=0} = h.$$

We call such a symmetric 2-form h an infinitesimal deformation. It is known that each infinitesimal deformation h satisfies

$$(*) \qquad \int_0^{2\pi} h(\dot{\gamma}(s), \dot{\gamma}(s))ds = 0$$

for any geodesic $\gamma(s)$ of (S^n, g_0) parametrized by arc-length (cf. [1] p. 151). V. Guillemin has proved in [2] that in the case of S^2 the condition $(*)$ is also sufficient for a symmetric 2-form h to be an infinitesimal deformation.

The purpose of this note is to show that the situation is completely different in the case of S^n $(n \geqq 3)$. We shall give another necessary condition for a symmetric 2-form h to be an infinitesimal deformation (Theorem 1). And we shall give a partial result for what h satisfies this condition (Propositions 2, 3).

2. We denote by $\mathcal{K}_2$ the vector space of symmetric 2-forms on S^n which satisfy $(*)$. Let $\sharp: T^*S^n \to TS^n$ be the bundle isomorphism defined by

$$g_0(\sharp(\lambda), v) = \lambda(v), \quad \lambda \in T_x^*S^n, \quad v \in T_xS^n, \quad x \in S^n.$$

Let E_0 be the function on T^*S^n such that

$$E_0(\lambda) = \frac{1}{2} g_0(\sharp(\lambda), \sharp(\lambda)), \qquad \lambda \in T^*S^n.$$

Consider the usual symplectic structure on T^*S^n, and let X_{E_0} be the symplectic vector field on T^*S^n defined by the hamiltonian E_0. E_0 and X_{E_0} are called the energy function and the geodesic flow associated with the metric g_0 respectively. We denote by $\{\xi_t\}$ the one-parameter group of transformations of T^*S^n generated by X_{E_0}. Then $\{\xi_t\}$ induces a free S^1-action of period 2π on the unit cotangent bundle S^*S^n. We define an operator $G: C^\infty(S^*S^n) \to C^\infty(S^*S^n)$ by

$$G(f)(\lambda)=\frac{1}{2\pi}\int_0^{2\pi} f(\xi_t\lambda)dt, \quad \lambda\in S^*S^n, \quad f\in C^\infty(S^*S^n).$$

Let $\tilde{\mathcal{H}}_2$ be the vector space of functions on T^*S^n which are quadratic forms on each fibre $T^*_xS^n$ ($x\in S^n$), and let $\mathcal{H}_2$ be the vector space of functions on S^*S^n which are the restrictions of elements of $\tilde{\mathcal{H}}_2$ onto S^*S^n. For each $h\in\mathcal{K}_2$ we define a function $\hat{h}$ on T^*S^n by

$$\hat{h}(\lambda)=h(\sharp(\lambda), \sharp(\lambda)), \qquad \lambda\in T^*S^n.$$

Moreover, let $X(h)$ be a homogeneous symplectic vector field on $T^*S^n\backslash\{\text{0-section}\}$ such that $X(h)E_0=\hat{h}$. We should remark that $X(h)$ exists for any $h\in\mathcal{K}_2$, but is not unique. We now define a symmetric bilinear map $F:\mathcal{K}_2\times\mathcal{K}_2\to C^\infty(S^*S^n)$ by

$$F(f,h)=G(X(f)\hat{h}), \qquad f, h\in\mathcal{K}_2.$$

It is easy to see that F is well-defined and is symmetric.

Then our first result is

Theorem 1. *Let $\{g_t\}$ be a one-parameter family of $C_{2\pi}$-metrics on S^n with g_0 being the standard one. Put $(d/dt)g_t|_{t=0}=h$. Then we have*

$$F(h,h)\in G(\mathcal{H}_2).$$

Remark. For S^2 it is known that $G(C^\infty(S^2)E_0)=G(\mathcal{H}_2)=$ Image of G. Thus the assertion of Theorem 1 has no meaning in this case.

The proof of Theorem 1 is based on the following lemma which is due to A. Weinstein (cf. [1] p. 122).

Lemma. *Let $\{g_t\}$ be as before, and let $\{E_t\}$ be the corresponding energy functions. Then there is a one-parameter family of homogeneous symplectic diffeomorphisms $\{\phi_t\}$ of $T^*S^n\backslash\{\text{0-section}\}$ such that ϕ_0 $=$ identity and $\phi_t^*E_0=E_t$.*

After differentiating both sides of the formula $\phi_t^*E_0=E_t$ two times in the variable t at $t=0$, we apply G to this formula. Then we have Theorem 1.

3. We shall give a partial result for what h satisfies the condition $F(h,h)\in G(\mathcal{H}_2)$. Consider S^n as the unit sphere in $\boldsymbol{R}^{n+1}$, and let $\iota: S^n\to\boldsymbol{R}^{n+1}$ be the inclusion. Let $x=(x_1,\cdots,x_{n+1})$ be the canonical coordinate functions on $\boldsymbol{R}^{n+1}$. Let P_m be the vector space of homogeneous polynomials $f(t,s)$ of degree m in two variables (t,s) whose degrees in the variable s are at most 1.

Proposition 2. *Consider a polynomial $f(x)$ of the form*

$$f(x)=f_1(x)+f_3(x)+\sum_{m=2}^{k} h_{2m+1}\left(\sum_{i=1}^{n+1} a_ix_i, \sum_{i=1}^{n+1} b_ix_i\right),$$

where $f_1(x)$ (resp. $f_3(x)$) is a polynomial of degree 1 (resp. degree 3) *in the variables $x=(x_1,\cdots,x_{n+1})$, $h_{2m+1}\in P_{2m+1}$, and a_i, b_i are real constants. Then we have*

$$F((\iota^*f)g_0, (\iota^*f)g_0)\in G(\mathcal{H}_2).$$

Proposition 3. *Let $f(x)$ be a homogeneous polynomial of degree*

$2k+1$ $(k\geqq 2)$ *in the variable* $x=(x_1, \cdots, x_{n+1})$. *Assume either* $f(x)$ *is a polynomial in only two variables* (x_1, x_2) *in case* $n\geqq 3$, *or each irreducible components of* $f(x)$ *in* $\boldsymbol{C}[x]$ *are also irreducible in* $\boldsymbol{C}[x]/(\sum_{i=1}^{n+1} x_i^2)$ *in case* $n\geqq 4$. *Suppose the symmetric* 2-*form* $(\iota^* f)g_0$ *on* S^n *satisfies the condition* $F((\iota^* f)g_0, (\iota^* f)g_0)\in G(\mathcal{H}_2)$. *Then there is a polynomial* $h(t, s)$ *in* P_{2k+1} *and real constants* a_i, b_i *such that* $f(x)=h(\sum_{i=1}^{n+1} a_i x_i, \sum_{i=1}^{n+1} b_i x_i)$.

For example, let $f(x)=x_1^{2k+1}+x_2^{2k+1}$ $(k\geqq 2)$. Then $(\iota^* f)g_0$ satisfies $(*)$. But it is clear that $f(x)$ cannot be written in the form $h(\sum_i a_i x_i, \sum_i b_i x_i)$ for any $h\in P_{2k+1}$. Therefore there is no $C_{2\pi}$-deformation $\{g_t\}$ of g_0 such that $(d/dt)g_t|_{t=0}=(\iota^* f)g_0$.

Remark. Let $f(x)$ be a polynomial of the form in Proposition 2 such that $f_3=0$ and (a_i) and (b_i) are linearly dependent. Then it is known that $(\iota^* f)g_0$ is really an infinitesimal deformation (Weinstein's example, cf. [1] p. 120). For any other case in Proposition 2 we do not know whether $(\iota^* f)g_0$ is an infinitesimal deformation or not.

The detailed proof will appear elsewhere.

References

[1] A. Besse: Manifolds All of Whose Geodesics are Closed. Springer-Verlag (1978).

[2] V. Guillemin: The Radon transforms on Zoll surfaces. Adv. in Math., **22**, 85–119 (1976).