20. *Depth of Rational Double Points on Quartic Surfaces*

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Introduction. The quartic surfaces in $CP^3:(x,y,z,w)$ with only rational double singular points are $K3$ surfaces of degree 4, when desingularized minimally. (But the converse is not true, which makes difficulty to the problem below.) Since the intersection form on the middle homology group has index $(+3,-19)$ for $K3$ surfaces, the sum of the ranks (the Milnor numbers) of singular points can not be greater than 19. If one restricts attention to a single rational double point on a quartic, one might naturally ask, how high the rank could be i.e. whether there would really be A_{19} or D_{19} on quartics. In this note we will answer it, giving the normal forms to the quartic surfaces with A_k (or D_k) singularity at a fixed point, say, $p=(0,0,0,1)$ for each k. Incidentally we prove also the rationality of components of the coarse moduli space of quartic surfaces with at least one A_k (or D_k) singular point for any k.

The existence resp. non-existence of A_{19} resp. D_{19} might be proved by the theory of period mapping combined with the classification of lattices of type A_{19} or D_{19} in the $K3$ lattice (Kulikov [2], Nikulin [3]; see also Saint-Donat [4], Shah [5]). Nonetheless it does not seem to be an obvious problem to prove the above rationality or to determine an explicit equation for the quartic with A_{19} on it and so on. This might justify this kind of work.

1. Case of A type. Every quartic surface in $CP^3:(x,y,z,w)$ with a double point at $p=(0,0,0,1)$ is written in the form

$$S:\ f(x,y,z)w^2+g(x,y,z)w+h(x,y,z)=0$$

where f, g, h are homogeneous polynomials of degree 2, 3, 4. If f, regarded as a quadratic form in x, y, z, is nondegenerate, then (S,p) is of type A_1; so we exclude this trivial case. But, if the rank of f is not greater than one, then (S,p) is not of A_k type for any k. Thus we assume that f is of rank 2. Then, by a suitable linear charge of x, y, z we can assume $f=xy$: now we will operate in the affine coordinates (x,y,z) by setting $w=1$. The equation of the surface S is:

$$F(x,y,z):=xy+g(x,y,z)+h(x,y,z)=0.$$

To recognize the type of singularity (S,p), we introduce the power

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series $\xi=\xi(z)$, $\eta=\eta(z)$ which are uniquely determined by the following condition:

$$(1)\qquad \begin{cases}(\partial F/\partial x)(\xi,\eta,z)=(\partial F/\partial y)(\xi,\eta,z)=0\\ \xi(0)=\eta(0)=\xi'(0)=\eta'(0)=0.\end{cases}$$

We set then $\tilde{F}(x,y,z)=F(x+\xi(z),\ y+\eta(z),\ z)$. Obviously $\tilde{F}=0$ defines an equivalent singularity at the origin; furthermore $\tilde{F}$ has the expansion of the following form $\tilde{F}=xy+F(\xi(z),\eta(z),z)+\cdots$ where dots indicate terms of weights higher than 1 with respect to weight system $(1/2, 1/2, 1/k)$ for any $k\geqq 3$; so, by the same reasoning as in [1], if we set

$$\textstyle\sum_\nu \Gamma_\nu z^\nu=F(\xi(z),\eta(z),z),$$

then we have:

Lemma. *(S,p) is a rational double point of type A_k if and only if $\Gamma_3=\cdots=\Gamma_k=0$, $\Gamma_{k+1}\neq 0$. If all Γ_k vanish, then the singular locus of S is not isolated near p.*

Γ_k $(k=3,4,\cdots)$ are polynomials of the coefficients of polynomials $g(x,y,z)$, $h(x,y,z)$, which are to be regarded as the parameters of the surface S. But we should note that we still have too many unnecessary variables. For, by using suitable projective transformations of CP^3, we can in fact kill some coefficients in g as follows: If $g(0,0,z)\neq 0$, then (S,p) is of type A_2; so we assume $g(0,0,z)=0$, excluding this trivial case. Thus g can be written in the form: $g_1(x,y)z^2+g_2(x,y)z+g_3(x,y)$. By a suitable scale change of x, y, z we can further assume that g_1 takes one of the following forms: $x+y$, x, 0. If $g=x+y$, we can then replace z by a suitable $z+ax+by$ so that g_2 is of the form cxy. But, then, replacing w by a suitable $w+ax+by+cz$, we can bring g into the form $(x+y)z^2+ax^3+by^3$ for some (other) a, b; and so on. We can *en resume* conclude that every quartic surface having an A_k singularity with $k\geqq 6$ at $p=(0,0,0,1)$ can be transformed into one of the following forms:

$$\text{(I)}\qquad F_1(x,y,z,w):=\{xw+z^2+by^2-x(z+\theta(x,y))\}\times\{yw+z^2+ax^2+y(z+\theta(x,y))\}+\phi(x,y)z+\psi(x,y)=0$$

$$\text{(II)}\qquad F_2(x,y,z,w):=\{xw+yz-x(bz+\theta(x,y))\}\times\{yw+z^2+ax^2+y(bz+\theta(x,y))\}+\phi(x,y)z+\psi(x,y)=0$$

$$\text{(III)}\qquad F_3(x,y,z,w):=\{xw+z^2+by^2-x\theta(x,y)\}\times\{yw+z^2+ax^2+y\theta(x,y)\}+\phi(x,y)z+\psi(x,y)=0$$

$$\text{(IV)}\qquad F_4(x,y,z,w):=\{xw+b_1y^2-x(b_2z+\theta(x,y))\}\times\{yw+z^2+ax^2+y(b_2z+\theta(x,y))\}+\phi(x,y)z+\psi(x,y)=0$$

$$\text{(V)}\qquad F_5(x,y,z,w):=xyw^2+(a_1x^2z+b_1y^2z+a_2x^3+b_2y^3)w+xz^3+b_3y^2z+\phi(x,y)z+\psi(x,y)=0$$

where $\theta(x,y)$, $\phi(x,y)$, $\psi(x,y)$ are homogeneous polynomials of degree 1, 3, 4. We regard each of these types as a family of quartic surfaces depending on the given parameters. (For example, for the family (I), the variables of the parameter space are just a, b and the coefficients of θ, ϕ, ψ.) We denote by $V(i)$ the parameter space of the family $F_i=0$. Now we can define series $\Gamma_\nu(i)$, $\nu=1, 2, \cdots$ of polynomials in the coordinates of $V(i)$ by setting $F(x, y, z)=F_i(x, y, z, 1)=xy+\cdots$ and defining power series $\xi(z)$, $\eta(z)$ just by (1) and setting $\sum_\nu \Gamma_\nu(i)z^\nu:=F(\xi, \eta, z)$. With Lemma in mind we introduce a descending chain of subvarieties:

$$V_k(i): \Gamma_7(i)=\cdots=\Gamma_k(i)=0 \qquad k=7, 8, \cdots.$$

We set:

$$V_\infty(i)=\cap_k V_k(i) \qquad V_k^*(i)=V_k(i)\setminus V_\infty(i)$$

$V_\infty(i)$ is just the set of points in the parameter space for which the surface $F_i=0$ has non-isolated singular locus near p. Among the varieties $V_k^*(i)$, some are irreducible and some are not. But at any rate we can show the components of $V_k^*(i)$ are always rational. We say that we give canonical forms to these surfaces if we choose an explicit system of generators of the function field for each of these components. We have done this by using the computer DEC 2020 (REDUCE 2). The computer first gives vast polynomials $\Gamma_{19}(1)$, $\Gamma_{18}(1)$ etc. But, for lower k, $\Gamma_k(i)$ are not so complicated, and we can go up just little by little by introducing, at each step k, suitable new parameters for $V_k^*(i)$ and eliminating one more parameters chosen for $V_{k-1}^*(i)$. The whole result is too complicated to be mentioned here. We only satisfy ourselves by summing it up in the following form:

Proposition 1. *For $i=3, 4$, $V_k^*(i)$ are empty if $k\geqq 16$. For $i=5$, $V_k^*(i)$ are empty if $k\geqq 12$. For $i=2$, only $V_{19}^*(i)$ is empty. $V_k^*(1)$ is an irreducible rational variety of dimension $19-k$ for $6\leqq k\leqq 16$ and for $k=18$. $V_{17}^*(1)$ has two irreducible components of dimension 2, both of which are rational. $V_{19}^*(1)$ consists of two points, which give projectively equivalent surfaces.*

Thus the surface with A_{19} is essentially unique. In fact it is given by the following equation:

$$16(x^2+y^2)+32xz^2-16y^3+16z^4-32yz^3+8(2x^2-2xy+5y^2)z^2$$
$$+8(2x^3-5x^2y-6xy^2-7y^3)z+20x^4+44x^3y+65x^2y^2+40xy^3+41y^4=0.$$

Remark 1. For the families (II), (III), (IV), (V) the automorphism group is not descrete while, for (I), there is only the involutive automorphism extending the projective transformation $x\leftrightarrow y$. In fact (II), (III), admit C^*-actions extending $(x, y, z)\to(t^3x, t^2y, tz)$, $(x, y, z)\to(t^2x, t^2y, tz)$ $(t\in C^*)$ respectively, and (IV), (V) admit $C^*\times C^*$-actions extending $(x, y, z)\to(sx, t^2y, tz)$ $(x, y, z)\to(sx, t^3y, tz)$ $((s, t)\in C^*\times C^*)$ respectively. For the family (I) we can prove that any two members are projectively equivalent if and only if they are transposed by the above involution. The other families we can also rigidify by fixing some suitable parameters to be constant so that they admit only a finite number of trivial automorphisms. When divided out by the automorphisms, they give some subsets to the coarse moduli space of quartic surfaces having at least one A_k-singular point for a lower k, whose dimensions we can count easily. Since this space is purely $19-k$ dimensional for each k it is easy to check whether the above subsets could really contribute as irreducible components of the moduli space.

2. Case of D type. In this case we may only treat one family of surfaces essentially:

$$F_d(x,y,z,w):=x^2w^2+\{(x+y)^2z+2xz^2\}w$$
$$+z^4+(x+2y)z^3+\left(\frac{1}{4}x^2+y\theta(x,y)\right)z^2+\phi(x,y)z+\psi(x,y)=0$$

where θ, ϕ, ψ are just as in the previous section. We regard the surface $S: F_d=0$ as a family of surfaces whose base space is parametrized by the coefficients of θ, ϕ, ψ. Now we set $\bar{F}(x,y,z):=z^{-2}F_d(xz,yz,z,1)$ and we introduce power series $\xi=\xi(z)$ and $\eta=\eta(z)$ by

$$\text{(2)} \qquad \begin{cases} \dfrac{\partial \bar{F}}{\partial x}(\xi,\eta,z)=\left(z^{-1}\dfrac{\partial \bar{F}}{\partial y}\right)(\xi,\eta,z)=0 \\ \xi(0)=\eta(0)=0. \end{cases}$$

We set further $\sum \Gamma_\nu(d)z^\nu:=F_d(z\xi(z),z\eta(z),z,1)$ to define polynomials $\Gamma_\nu(d)$ of the coordinates of the parameter space. Just as in the former case we have: *(S,p) is of type D_k if and only if $\Gamma_7(d)=\cdots=\Gamma_{k-2}(d)=0$, $\Gamma_{k-1}(d)\neq 0$. If all $\Gamma_k(d)$ vanish, then the singular locus of S is not isolated near p.*

We introduce as before the following subvarieties in the parameter space:

$$V_k(d): \Gamma_7(d)=\cdots=\Gamma_{k-2}(d)=0$$
$$V_\infty(d):=\bigcap_k V_k(d) \qquad V_k^*(d):=V_k(d)\setminus V_\infty(d).$$

Now the main result for this case is stated as follows:

Proposition 2. *For $k\neq 12,16,19$, $V_k^*(d)$ is an irreducible rational variety of dimension $19-k$, while for $k=12,16$ it has two irreducible components which are both $(19-k)$-dimensional and rational. $V_{19}^*(d)$ is empty.*

References

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