19. A Note on Refinable Maps and Quasi-Homeomorphic Compacta

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It is assumed that all spaces are metrizable and maps are continuous. A connected compactum is a continuum. A map $f: X\to Y$ between compacta is said to be an *ε-mapping* if f is surjective and $\operatorname{diam} f^{-1}(y)<\varepsilon$ for each $y\in Y$. A compactum X is *Y-like* if for each $\varepsilon>0$ there is an ε-mapping X to Y. Two compacta X and Y are *quasi-homeomorphic* [2] if X is Y-like and Y is X-like. A map $r: X\to Y$ between compacta is *refinable* [5] if for each $\varepsilon>0$ there is an ε-mapping $f: X\to Y$ such that $d(r,f)=\sup\{d(r(x),f(x))\mid x\in X\}<\varepsilon$.

In [6], H. Roslaniec proved the following

Theorem (H. Roslaniec). *If X and Y are quasi-homeomorphic compact subsets of the Euclidean n-dimensional space E^n, then E^n-X and E^n-Y have the same number of components.*

In [3] and [4], the author investigated shape theoretic properties of refinable maps. One of the purposes of this note is to prove the following

Theorem 1. *If X and Y are compact subsets of E^n and admit a refinable map $r: X\to Y$, then E^n-X and E^n-Y have the same number of components.*

Corollary 2 ([3, Corollary 2.5]). *If X and Y are continua contained in the plane E^2 and admit a refinable map $r: X\to Y$, then X and Y have the same shape*, i.e., $\mathrm{Sh}(X)=\mathrm{Sh}(Y)$.

In [2], K. Borsuk showed that there exist quasi-homeomorphic compacta (non connected) X, $Y\subset E^3$ such that $\mathrm{Sh}(X)\neq\mathrm{Sh}(Y)$. H. Roslaniec [6] asked the following question: Is it true that quasi-homeomorphic continua have the same shape? We show that the question has a negative answer. In Example 5 (below), we give Peano continua X and Y in E^3 such that (1) $\mathrm{Sh}(X)\neq\mathrm{Sh}(Y)$, (2) X and Y are quasi-homeomorphic and (3) there is a refinable map $r: X\to Y$.

To prove Theorem 1, we will use the following

Lemma 3 ([3, Theorem 1.5]). *If a map $r: X\to Y$ between compacta is refinable, then r induces a pseudo-isomorphism in shape category.*

Lemma 4 ([6, Lemma 1]). *Let X be a compact subset of E^n and U be a neighborhood of X in E^n. Then there is a compact polyhedron W*

such that (1) $X \subset \mathrm{Int}\, W \subset W \subset U$, *and* W *satisfies one of the following conditions* (2) *and* (2)′.

(2) *If* $E^n - X$ *has* $m\,(<\infty)$ *components* $S_1, S_2, \cdots S_m$, *then* $E^n - W$ *has also* m *components* $T_1, T_2, \cdots, T_m$ *such that* $T_i \subset S_i$ $(i=1,2,\cdots,m)$.

(2)′ *If* $E^n - X$ *has* ∞ *components* $S_1, S_2, \cdots$, *then for each* j *there is at most one component of* $E^n - W$ *which is contained in* S_j. *Moreover, it can be assumed that* $E^n - W$ *has more components than any fixed natural number* k. *A polyhedron which satisfies* (1) *and* (2) *will be denoted by* $W(U,X)$. *A polyhedron which satisfies* (1) *and* (2)′ *will be denoted by* $W(U,X;k)$.

Proof of Theorem 1. First, we shall prove that the number of components of $E^n - X$ is not less than the number of components of $E^n - Y$. We may assume that $E^n - X$ has $m_1\,(<\infty)$ components. Suppose, on the contrary, that $E^n - Y$ has $m_2\ (>m_1)$ components. If $m_2<\infty$, let $W_2 = \mathrm{W}(E^n, Y)$. If $m_2=\infty$, let $W_2 = W(E^n, Y; m_1+1)$. Since E^n is an AR, there is an extension $R: E^n \to E^n$ of $r: X \to Y$. By Lemma 4, there is a compact polyhedron $W_1 = W(R^{-1}(W_2), X)$. By Lemma 3, r induces a pseudo-isomorphism in shape category. Hence there is a compact polyhedron $W_3 \subset W_2$ and a map $g: W_3 \to W_1$ such that $Rg \simeq i$ in W_2, where $i: W_3 \to W_2$ is the inclusion. Moreover, we may assume that if $m_2<\infty$, $W_3 = (W_2, Y)$, and if $m_2 = \infty$, $W_3 = W(W_2, Y; m_1+1)$. Consider the following commutative diagram (see [7]),

$$\begin{array}{ccccc} H^{n-1}(W_3) & \xleftarrow{\varphi_{W_3}} & H_1(E^n, E^n - W_3) & \xrightarrow{\partial_{W_3}} & \tilde{H}_0(E^n - W_3) \\ \uparrow{\scriptstyle H^{n-1}(i)} & & \uparrow{\scriptstyle H_1(j)} & & \uparrow{\scriptstyle \tilde{H}_0(j)} \\ H^{n-1}(W_2) & \xleftarrow[\varphi_{W_2}]{} & H_1(E^n, E^n - W_2) & \xrightarrow[\partial_{W_2}]{} & \tilde{H}_0(E^n - W_2) \end{array}$$

where $j: E^n - W_2 \to E^n - W_3$ is the inclusion and H_* and H^* denote the singular homology and cohomology with coefficients in integers Z, respectively. Note that φ_{W_3}, φ_{W_2}, ∂_{W_3} and ∂_{W_2} are isomorphisms. By the choice of W_3, $\tilde{H}_0(j)$ is a monomorphism, hence $H^{n-1}(i)$ is also a monomorphism. Consider the following commutative diagram

$$\begin{array}{ccc} & \xrightarrow{H^{n-1}(g)} & H^{n-1}(W_3) \\ & & \uparrow{\scriptstyle H^{n-1}(i)} \\ H^{n-1}(W_1) & \xleftarrow[H^{n-1}(R\,|\,W_1)]{} & H^{n-1}(W_2) \end{array} .$$

Note that $H^{n-1}(W_2) \supset Z^{m_1}$ and $H^{n-1}(W_1) = Z^{m_1-1}$. Since $H^{n-1}(i)$ is a monomorphism, $H^{n-1}(i)(H^{n-1}(W_2)) \supset Z^{m_1}$. This implies the contradiction. The converse is similar.

Example 5. Consider the following set in E^3.

$K_n = \{(x, y, 0) \in E^3 \,|\, (x-(2n+1)/4n(n+1))^2 + (y-(2n+1)/4n(n+1))^2 < (1/4n(n+1))^2\}$, $\quad (n=1,2,\cdots)$.

$A_1 = D - \bigcup_{n=1}^{\infty} K_n$, where D denotes the subset in the plane $z=0$ which is the triangle with vertices $(0,0,0)$, $(1,0,0)$ and $(0,1,0)$.

$A_2=\{(x, y, z)\in E^3 \mid (x, y, 0)\in A_1$ and $-(x+y)\leqq z\leqq x+y\}$.

$B=\mathrm{Bd}_{E^3}A_2$ (see [1]).

Then B is a 2-dimensional Peano continuum and not movable (see [1]). There is an inverse sequence $\underline{B}=\{(B_n, b_n), p_{n,n+1}\}$ such that $(B, (0, 0, 0)) = \mathrm{invlim}\,\underline{B}$, $p_{n,n+1}: (B_{n+1}, b_{n+1})\to(B_n, b_n)$ is surjective and each B_n is a closed surface with genus n. By identifying the points $b_1, b_2, \cdots$ of $B_1, B_2, \cdots$ we obtain a continuum $(Y, *)=\bigvee_{n=1}^{\infty}(B_n, b_n)$ with a metric d_Y on Y such that $d_Y(x, y)<1/n$ if $x, y\in B_n$. Then Y is a Peano continuum which is homeomorphic to a compact subset of E^3. Similarly, we obtain a Peano continuum $(X, *)=(B, (0, 0, 0))\vee(Y, *)$ by identifying the points $(0, 0, 0)$ and $*$. Note that X is homeomorphic to a compact subset of E^3. Define a map $r: X\to Y$ by $r(x)=x$ if $x\in Y$, $r(x)=*$ if $x\in B$. Then r is refinable (cf. [3, Example 2.6]). In particular, X is Y-like. Next we show that Y is X-like. Let $\varepsilon>0$. Choose a number m with $\varepsilon>1/m$. Since $B\cup B_m$ $(\subset X)$ is a Peano continuum, by Hahn-Mazurkiewicz's theorem, there is an onto map $g_m: (B_m, b_m)\to(B\cup B_m, *)$. Define a map $g: Y\to X$ by

$$g(y)=\begin{cases} y, & \text{if } y\in\bigcup_{n=1}^{m-1}B_n\cup\bigcup_{n=m+1}^{\infty}B_n,\\ g_m(y), & \text{if } y\in B_m.\end{cases}$$

Then g is an ε-mapping, which implies that Y is X-like. Since B $(\subset X)$ is a retract of X and B is not movable, X is not movable. On the other hand, Y is movable. Hence $\mathrm{Sh}(X)\neq\mathrm{Sh}(Y)$.

References

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