# 18. On a Pseudo-Runge-Kutta Method of Order 6

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**1. Introduction.** The present paper is concerned with the numerical solution of the initial value problem:

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0. \end{cases} \tag{1.1}$$

In his paper [6], the author has proposed some new method of Runge-Kutta type, and we have seen that there exists an $r$-stage Pseudo-Runge-Kutta method of order $r+2$ for $r=2, 3$.

Recently he has derived some improvements for this. See [7].

In this paper we shall give a four-stage method of order six based on this improved method. We include also a numerical result to compare our formula with the Huta formula of order 6.

Detailed proofs and related results will appear elsewhere.

**2. Numerical method.** We consider the Pseudo-Runge-Kutta method, i.e.

$$\begin{aligned} &y_{n+1} = y_n + v(y_n - y_{n-1}) + h\Phi(x_{n-1}, x_n, y_{n-1}, y_n; h) \\ &\Phi(x_{n-1}, x_n, y_{n-1}, y_n; h) = \sum_{i=0}^{4} w_i k_i, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} &k_0 = f(x_{n-1}, y_{n-1}) \qquad k_1 = f(x_n, y_n), \\ &k_2 = f\left(x_n + a_2 h, y_n + b_0(y_n - y_{n-1}) + h \sum_{i=1}^{2} b_i k_i\right), \\ &k_3 = f\left(x_n + a_3 h, y_n + c_0(y_n - y_{n-1}) + h \sum_{i=1}^{3} c_i k_i\right), \\ &k_4 = f\left(x_n + a_4 h, y_n + d_0(y_n - y_{n-1}) + h \sum_{i=1}^{4} d_i k_i\right). \end{aligned}$$

In the above formula (2.1), the value $y_n$ is to be an approximation to the value $y(x_n)$ of the solution of (1.1) for $x_n = x_0 + nh$.

Throughout the paper, the coefficients $a_i (i=1, 2, 3, 4)$, $b_i (i=0, 1, 2)$, $c_i (i=0, 1, 2, 3)$ and $d_i (i=0, 1, 2, 3, 4)$ are constrained by the following conditions:

$$a_2 = \sum_{i=0}^{2} b_i, \quad a_3 = \sum_{i=0}^{3} c_i, \quad a_4 = \sum_{i=0}^{4} d_i \quad (0 \leqq a_2, a_3, a_4 \leqq 1). \tag{2.2}$$

Assume that $y_n - z(x_n) = O(h^7)$, where $z(x)$ is the solution of the initial value problem $z' = f(x, z)$, $z(x_{n-1}) = y_{n-1}$. Using the same notations as in [6], Taylor expansion for (2.1) is

$$
\begin{aligned}
y_{n+1}= & y_n+hA_1k_1+h^2A_2T+\frac{1}{2!}h^3(A_3f_yT+A_4T^2) \\
& +\frac{1}{3!}h^4(B_1T^3+B_2f_yT^2+B_3f_y^2T+3B_4ST) \\
& +\frac{1}{4!}h^5(C_1T^4+6C_2TS^2+4C_3T^2S+3C_4f_{yy}Q+C_5f_yT^3 \\
& \qquad +C_6f_y^2T^2+C_7f_y^3T+C_8f_yTS) \\
& +\frac{1}{5!}h^6(D_1T^5+D_2TS^3+D_3T^2S^2+D_4T^3S+D_5f_{yy}T^2T \\
& \qquad +D_6QR+D_7TP+D_8f_yT^4+D_9f_y^2T^3+D_{10}f_y^3T+D_{11}f_y^4T \\
& \qquad +D_{12}f_{yy}f_yQ+D_{13}f_yTS^2+D_{14}f_y^2TS+D_{15}f_yT^2S)+O(h^7),
\end{aligned}
$$

where $\{A_i\}$, $\{B_i\}$, $\{C_i\}$ and $\{D_i\}$ are some constants to be determined.

The method (2.1) is of order 6 if

$$
\begin{aligned}
(2.3)\qquad & A_1=1,\quad A_2=\frac{1}{2},\quad A_3=A_4=\frac{1}{3},\quad B_i=\frac{1}{4}\,(i=1,2,3,4), \\
& C_i=\frac{1}{5}\,(i=1,\cdots,7),\quad D_i=\frac{1}{6}\,(i=1,\cdots,7,12,\cdots 15), \\
& D_8=\frac{8}{6},\quad D_9=\frac{8}{3},\quad D_{10}=2,\quad D_{11}=\frac{3}{2}.
\end{aligned}
$$

We consider the case $w_2=0$. From (2.3) we have

$$
\begin{aligned}
(2.4)\qquad & a_3=\frac{1}{\sqrt{3}},\quad a_4=1,\quad v=\frac{1}{11}(139-80\sqrt{3}), \\
& w_0=\frac{1}{33}(54-31\sqrt{3}),\quad w_3=\frac{6}{11}(15-8\sqrt{3}), \\
& w_4=\frac{1}{33}(6-\sqrt{3}),\quad b_0=-(2a_2^3+3a_2^2), \\
& b_1=a_2^3+a_2^2,\quad b_2=a_2(a_2+1)^2,\quad c_0=\frac{2(2+\sqrt{3})}{3(2a_2+1)}-\frac{2\sqrt{3}}{9}-1, \\
& c_1=\frac{-(2+\sqrt{3})(2+3a_2)}{9a_2(2a_2+1)(a_2+1)}+\frac{\sqrt{3}}{9}+\frac{1}{3}, \\
& c_2=\frac{(2+\sqrt{3})(1+3a_2)}{9a_2(2a_2+1)}+\frac{4\sqrt{3}}{9}+\frac{2}{3},\quad c_3=\frac{2+\sqrt{3}}{9a_2(2a_2^2+3a_2+1)}, \\
& d_0=6\left(8\sqrt{3}-\frac{77}{6}-\frac{2}{2a_2+1}\right),\quad d_1=\frac{6a_2+4}{2a_2^3+3a_2^2+1}+8-6\sqrt{3}, \\
& d_2=16-12\sqrt{3}+\frac{12}{2a_2+1}-\frac{6a_2^2+4a_2-2}{2a_2^3+3a_2^2+a_2}, \\
& d_3=\frac{-2}{2a_2^3+3a_2^2+a_2},\quad d_4=54\left(1-\frac{5}{9}\sqrt{3}\right).
\end{aligned}
$$

**3. Computational results.** Computations are done in double precision arithemetic on the FACOM M-200 of Kyushu University.

The following is a comparison of the errors incurred by using the sixth order Huta method and the method (2.4) with various $a_2$ to solve $y'=-y+x^2$, $y(0)=3$, whose solution is $y(x)=\exp(-x)+2-2x+x^2$. $h=1/2^4$.

| | $x=2.00$ | $x=4.00$ | $x=6.00$ |
|---|---|---|---|
| Exact solution | $0.2135E+1$ | $0.1101E+2$ | $0.2600E+2$ |
| Error for Huta method | $0.7703E-11$ | $0.8358E-11$ | $0.8412E-11$ |
| Frror for the formula (2.4) | | | |
| $a_2=0.7$ | $-0.2756E-10$ | $-0.7578E-11$ | $-0.1442E-11$ |
| $a_2=0.5$ | $-0.2631E-10$ | $-0.7235E-11$ | $-0.1367E-11$ |
| $a_2=0.3$ | $-0.2443E-10$ | $-0.6721E-11$ | $-0.1261E-11$ |

**Remarks.** (1) In the table, the value $y_1$ necessary for the evaluation using the formula (2.4) is computed by Huta formulas.

(2) The Huta formula is of eight stage step process, while ours is of four stage step process.

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