17. A Calculus of the Gauss-Manin System of Type A_l. II

The Hamiltonian Representation

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The present note is the later half of our article titled "A calculus of the Gauss-Manin system of type A_l". We keep the notation and the terminology in our previous note [4].

3. **The flat coordinate system.** Now we return to the setting of no. 1 and work with the ring $R((x^{-1}))$, where $R=\mathbf{C}[s_2, s_3, \cdots]$. We define a new "coordinate system" $(z_2, z_3, \cdots)$ for R in place of $(s_2, s_3, \cdots)$ by the formula

(3.1) $$x=f-\sum_{i=2}^{\infty} z_i f^{1-i}.$$

It is easy to see that $z_2, z_3, \cdots$ are determined inductively as polynomials in $s_2, s_3, \cdots$ and satisfy $\partial_{s_i}(z_i)=1$ and $\partial_{s_j}(z_i)=0$ for $i<j$. The sequence $(z_2, z_3, \cdots)$ in R will be called the flat coordinates associated with $(s_2, s_3, \cdots)$.

Theorem 2 (Versality formula). *The flat coordinate system $(z_2, z_3, \cdots)$ is characterized by the formulae*

(3.2) $$\partial_{z_j}(f)=\partial_x(f)f^{1-j} \qquad \text{for } j=2,3,\cdots.$$

Moreover we have

(3.3) $$\partial_{z_j}(F_k)=ke_{k-j} \qquad \text{for } j=2,3,\cdots.$$

For an indeterminate u, set $s(u)=\sum_{i=2}^{\infty} s_i u^i$ and $z(u)=\sum_{i=2}^{\infty} z_i u^i$. Then the coordinate transformations between the two coordinate systems are given by

(3.4) $$z_j=\frac{1}{j-1}(1+s(u))_j^{j-1} \qquad \text{for } j=2,3,\cdots$$

and

(3.5) $$s_j=\frac{-1}{j-1}(1-z(u))_j^{j-1} \qquad \text{for } j=2,3,\cdots.$$

An advantage of our formation of the flat coordinates lies in the following two theorems, which will play an essential role in no. 4.

Theorem 3 (Elimination of the variable x). *In terms of the flat basis $(e_k)_{k\in\mathbf{N}}$ for $R[x]$, the flat coordinates $(z_2, z_3, \cdots)$ are represented by*

(3.6) $$z_j=-(1+e(u))_j^{-1} \qquad \text{for } j=2,3,\cdots,$$

where $e(u)=\sum_{i=1}^{\infty} e_i u^i$.

Theorem 4. *The sequence $(F_k)_{k\in\mathbf{N}}$ is represented by*

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(3.7) $\qquad F_k = k\log(1+e(u))_k \qquad$ *for* $k=1,2,\cdots,$

where $\log(1+e(u))_k$ *stands for the coefficient of* u^k *in the Taylor expansion of* $\log(1+e(u))$.

By the l-reduction $R \to R_l$, $z_2, z_3, \cdots$ define a sequence in $R_l = \mathbf{C}[t_2, \cdots, t_l]$. Set

(3.8) $\qquad y_i = l z_i \qquad$ for $i=2,\cdots,l$

in R_l. Then $(y_2,\cdots,y_l)$ coincides with the "flat generator system" of type A_{l-1} in the sense of K. Saito, T. Yano and J. Sekiguchi [2]. We call the sequence $(y_2,\cdots,y_l)$ the flat coordinates associated with $(t_2,\cdots,t_l)$. Then, for the versal deformation $F=x^l+t_2x^{l-2}+\cdots+t_l$ of type A_{l-1}, we have

(3.9) $\qquad \partial_{y_j}(F) = e_{l-j} \qquad$ for $j=2,\cdots,l$ in $R_l[x]$.

Note also that $(1/l)\partial_x(F)=e_{l-1}$. The coordinate transformations between $(t_2,\cdots,t_l)$ and $(y_2,\cdots,y_l)$ are given by the following:

(3.10) $$y_j = \frac{l}{j-1}(1+t(u))_j^{(j-1)/l} \qquad \text{for } j=2,\cdots,l$$

and

(3.11) $$\begin{cases} t_j = \dfrac{-l}{j-l}\left(1-\dfrac{1}{l}\,y(u)\right)_j^{j-l} & \text{for } j=2,\cdots,l-1,\\ t_l = -l\log\left(1-\dfrac{1}{l}\,y(u)\right)_l, \end{cases}$$

where $y(u)=\sum_{i=2}^{l} y_i u^i$.

4. The Hamiltonian representation and a quantized contact transformation. In what follows, we give a canonical representation of the Gauss-Manin system H_F associated with the versal deformation $F=x^l+t_2x^{l-2}+\cdots+t_l$ of type A_{l-1} $(l\geq 2)$. By doing so, we can determine the quantized contact transformation which reduces H_F to a standard form.

Let $(y_2,\cdots,y_l)$ be the flat coordinates associated with $(t_2,\cdots,t_l)$ (no. 3). Then, by the versality formula (3.9), we get

(4.1) $\qquad \partial_{y_{l-i}}\partial_{y_l}^{-1}\delta(F) = e_i\delta(F) \qquad$ for $i=0,\cdots,l-2$ in $\mathcal{C}_{[F]}$.

We set $\partial_{i*}=\partial_{y_{i*}}$, where $i^*=l-i$, for $i=0,\cdots,l-2$.

Proposition 4. (i) *Let i and j be integers with $1\leq i, j\leq l-2$. Then we have*

(4.2) $$\partial_{i*}\partial_{j*}\partial_{0*}^{-2}\delta(F) = \begin{cases} e_ie_j\delta(F) & \text{if } i+j<l,\\ e_ie_j\delta(F)+\dfrac{1}{l}\partial_{0*}^{-1}\delta(F) & \text{if } i+j=l. \end{cases}$$

(ii) *Let $\alpha=(\alpha_1,\cdots,\alpha_{l-2})\in\mathbf{N}^{l-2}$ be a multi-index such that*

$$|\alpha| = \sum_{i=1}^{l-2}\alpha_i \geq 3 \quad \textit{and} \quad \|\alpha\| = \sum_{i=1}^{l-2} i\alpha_i \leq l.$$

Then we have

(4.3) $$\partial_{1*}^{\alpha_1}\cdots\partial_{(l-2)*}^{\alpha_{l-2}}\partial_{0*}^{-|\alpha|}\delta(F) = e_1^{\alpha_1}\cdots e_{l-2}^{\alpha_{l-2}}\delta(F).$$

We denote by η_j the covector corresponding to the operator $\partial_j=\partial_{y_j}$ for $j=2, \cdots, l$ and set

$$H(\eta)=H(\eta_2, \cdots, \eta_l) \tag{4.4}$$
$$=-l\eta_l \log\left(1+\frac{1}{\eta_l}\eta_*(u)\right)_l,$$

where $\eta_*(u)=\sum_{i=1}^{l-2}\eta_{i*}u^i$, and

(4.5) $\quad H'_j(\eta)=\partial_{\eta_j}H(\eta) \qquad$ for $j=2, \cdots, l$.

Thus we obtain a sequence $H'_2(\partial_y), \cdots, H'_l(\partial_y)$ of micro-differential operators in $\mathcal{E}_{S,(0,dy_l)}$:

(4.6) $\quad H'_j(\partial_y)=-l(1+\partial_l^{-1}\partial_*(u))_j^{-1} \qquad$ for $j=2, \cdots, l-1$

and

(4.7) $\quad H'_l(\partial_y)=-l(1+\partial_l^{-1}\partial_*(u))_l^{-1}-l\log(1+\partial_l^{-1}\partial_*(u))_l,$

where $\partial_*(u)=\sum_{i=1}^{l-2}\partial_{i*}u^i$. By Theorems 3 and 4 combined with Proposition 4, we obtain

Theorem 5 (Hamiltonian representation for H_F). *With the notations above, the Gauss-Manin system H_F of type A_{l-1} is represented as the following system of micro-differential equations:*

$$\begin{cases} y_j u=H'_j(\partial_y)u & \textit{for } j=2, \cdots, l-1 \quad \textit{and} \\ y_l u=H'_l(\partial_y)u+\dfrac{l-1}{2}\partial_{y_l}^{-1}u. \end{cases} \tag{4.8}$$

In other words, we have an isomorphism

$$\mathcal{E}(0)\Big/\sum_{j=2}^{l}\mathcal{E}(0)P_j \xrightarrow{\sim} H_F^{(0)},$$

where $\mathcal{E}(0)=\mathcal{E}_S(0)_{(0,dy_l)}$ and

$$P_j=y_j-H'_j(\partial_y) \qquad \text{for } j=2, \cdots, l-1,$$
$$P_l=y_l-H'_l(\partial_y)-\frac{l-1}{2}\partial_{y_l}^{-1}. \tag{4.9}$$

Corollary. *With the coordinates $(y_2, \cdots, y_l; \eta_2, \cdots, \eta_l)$ of T^*S, the characteristic variety of the Gauss-Manin system H_F is defined by the equations*

(4.10) $\quad y_j=H'_j(\eta) \qquad$ *for* $j=2, \cdots, l,$

near the codirection $(0, dy_l)$.

Note that the equations (4.10) give a parametrization of the discriminant set of the versal deformation F.

Let $T=\mathbf{C}^{l-1}$ be another complex affine $(l-1)$-space with coordinates $(x_2, \cdots, x_l)$. We define a quantized contact transformation

$$\Phi: \mathcal{E}_{S,(0,dy_l)} \xrightarrow{\sim} \mathcal{E}_{T,(0,dx_l)}$$

as follows. (For the generalities of quantized contact transformations, see F. Pham [1].) Set

(4.11) $\quad h(x_2, \cdots, x_{l-1})=-l\log(1+x_*(u))_l,$

where $x_*(u)=\sum_{i=1}^{l-2}x_{i*}u^i$. As the kernel form of the transformation Φ, we take

(4.12) $\gamma=\delta(y_l-x_l-h(x_2,\cdots,x_{l-1})-\sum_{i=2}^{l-1}y_ix_i)\otimes dy_2\wedge\cdots\wedge dy_{l-1}$.

Then the transformation Φ is defined by

$$\Phi(P)\cdot\gamma=\gamma\cdot P \qquad \text{for each } P\in\mathcal{E}_{S,(0,dy_l)}.$$

Theorem 6. *By the quantized contact transformation Φ with kernel form γ, the Gauss-Manin system H_F is transformed to the following system of micro-differential equations for $\delta^{((1-l)/2)}(x_l)$:*

$$(4.13)\qquad \begin{cases}\partial_{x_j}\partial_{x_l}^{-1}u=0 & \text{for } j=2,\cdots,l-1 \text{ and}\\ x_lu=\dfrac{l-1}{2}\partial_{x_l}^{-1}u.\end{cases}$$

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