15. *Generalized Fuglede-Putnam Theorem and Hilbert-Schmidt Norm Inequality*[*)]

By Takayuki FURUTA
Department of Mathematics, Faculty of Science, Hirosaki University

(Communicated by Kôsaku YOSIDA, M. J. A., Feb. 12, 1982)

1. Introduction. An operator means a bounded linear operator on a separable Hilbert space H. An operator T is called *quasinormal* if T commutes with T^*T, *subnormal* if T has a normal extension and *hyponormal* if $[T^*, T] \geqq 0$ where $[S, T] = ST - TS$. The inclusion relation of these classes of non-normal operators listed above is as follows:

$$\text{Normal} \subsetneqq \text{Quasinormal} \subsetneqq \text{Subnormal} \subsetneqq \text{Hyponormal}$$

the above inclusions are all proper [7, Problem 160, p. 101].

The familiar Fuglede-Putnam theorem asserts that $AX = XB$ implies $A^*X = XB^*$ when A and B are normal operators [1] [8] [9]. As a generalized version of this Fuglede-Putnam theorem, we show that let A and B^* be hyponormal and let C be hyponormal commuting with A^* and also let D^* be hyponormal commuting with B respectively, then for every Hilbert-Schmidt operator X, the Hilbert-Schmidt norm of $AXD + CXB$ is greater than or equal to the Hilbert-Schmidt norm of $A^*XD^* + C^*XB^*$. In particular, $AXD = CXB$ implies $A^*XD^* = C^*XB^*$. If we strengthen the hyponormality conditions on A, B^*, C and D^* to quasinormality, we can relax Hilbert-Schmidt operator of the hypothesis on X to be every operator and still retain the inequality under suitable hypotheses.

In this paper we show Theorem 1 and also Theorem 2 by integrating the results in [2]–[5] and [10].

2. Statement of results. Theorem 1. *Let A and B^* be hyponormal on H. Let C be hyponormal commuting with A^* and also D^* be hyponormal commuting with B respectively. Then*

(i) (*) $\quad \|AXD + CXB\|_2 \geqq \|A^*XD^* + C^*XB^*\|_2$

holds for every X in Hilbert-Schmidt class. Equality in (*) holds for every X in Hilbert-Schmidt class when A, B, C and D are all normal.

(ii) *If X is an operator in Hilbert-Schmidt class such that $AXD = CXB$, then $A^*XD^* = C^*XB^*$.*

Corollary 1. *Let A and B^* be hyponormal on H. Let C be normal commuting with A and also let D be normal commuting with B*

---

*) Dedicated in deep sorrow to the memory of the late Professor Teishirô Saitô.

respectively. Then

(i) (*) $\|AXD+CXB\|_2 \geqq \|A^*XD^*+C^*XB^*\|_2$

holds for every X in Hilbert-Schmidt class. Equality in (*) *holds for every X in Hilbert-Schmidt class when A and B are both normal.*

(ii) *If X is an operator in Hilbert-Schmidt class such that* $AXD=CXB$, *then* $A^*XD^*=C^*XB^*$.

Definition 1. Let $[S, T]_*$ denote the following "**-commutator*":

$$[S, T]_* = ST - TS^*$$

this *-commutator is completely different from usual commutator $[S, T]$.

Definition 2. Let S_T denote the positive square root of $[T^*, T]$ for hyponormal operator T.

Theorem 2. *Let A and* B^* *be quasinormal on H. Let C be quasinormal such that commutes with A and satisfies* $[A, S_C]_* = [C, S_A]_*$ *and also let* D^* *be quasinormal such that commutes with* B^* *and satisfies* $[B^*, S_{D^*}]_* = [D^*, S_{B^*}]_*$ *respectively. Then*

(i) (**) $\|AXD+CXB\|_2 \geqq \|A^*XD^*+C^*XB^*\|_2$

holds for every X in B(H). Equality in (**) *holds for every X in B(H) when A, B, C and D are all normal.*

(ii) *If X is an operator such that* $AXD=CXB$, *then* $A^*XD^*=C^*XB^*$.

Corollary 2. *Let A and* B^* *be quasinormal on H. Let C be normal commuting with A and also D be normal commuting with B respectively. Then*

(i) (**) $\|AXD+CXB\|_2 \geqq \|A^*XD^*+C^*XB^*\|_2$

holds for every X in B(H). Equality in (**) *holds for every X in B(H) when A, B, C and D are all normal.*

(ii) *If X is an operator such that* $AXD=CXB$, *then* $A^*XD^*=C^*XB^*$.

Remark. If we strengthen on X to be in Hilbert-Schmidt class in Corollary 2, then we can relax quasinormality of the hypotheses on A and B^* to hyponormality and still retain the inequality, that is, just Corollary 1.

Proofs and details will appear in [5] together with some results. This paper is closely related to Goya and Saitô [6]. Here the author would like to dedicate in much sorrow to the late Prof. Teishirô Saitô and he should like to read mass for the repose of his soul.

References

[1] S. K. Berberian: Note on a theorem of Fuglede and Putnam. Proc. Amer. Math. Soc., **10**, 175–182 (1959).

[2] T. Furuta: Relaxation of normality in the Fuglede-Putnam theorem. ibid., **77**, 324–328 (1979).

[3] T. Furuta: Normality can be relaxed in the asymptotic Fuglede-Putnam theorem. Proc. Amer. Math. Soc., **79**, 593–596 (1980).
[4] ——: An extension of the Fuglede-Putnam theorem to subnormal operators using a Hilbert-Schmidt norm inequality. ibid., **81**, 240–242 (1981).
[5] ——: A Hilbert-Schmidt norm inequality associated with the Fuglede-Putnam theorem (to appear in Bull. Australian Math. Soc.).
[6] E. Goya and T. Saitô: On intertwining by an operator having a dense range. Tohoku Math. Journ., **33**, 127–131 (1981).
[7] P. R. Halmos: A Hilbert Space Problem Book. Van Nostrand, Princeton, N. J. (1967).
[8] C. R. Putnam: On normal operators in Hilbert space. Amer. J. Math., **73**, 357–362 (1951).
[9] M. Rosenblum: On a theorem of Fuglede and Putnam. J. London Math. Soc., **33**, 376–377 (1958).
[10] G. Weiss: Fuglede's commutativity theorem modulo the Hilbert-Schmidt class and generating functions for matrix operators. II. J. Operator Theory, **5**, 3–16 (1981).