14. *Analytic Hypo-Ellipticity and Propagation of Regularity for Operators with Non-Involutory Characteristics*

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We study matrices of microdifferential operators of the form $P=P_1P_2I_m+Q$; here P_1 and P_2 are scalar operators such that the Poisson bracket of their principal symbols never vanishes, Q is an $m\times m$ matrix of operators of lower order, and I_m denotes the unit matrix of degree m.

In § 1, we study the propagation of micro-analyticity of solutions of the equation $Pu=0$ when the principal symbol of P_1 is real. Theorem 1 is a partial generalization of Corollary 3.7 of [3], where the principal symbol of P_2 was also assumed to be real.

In § 2, we study the analytic hypo-ellipticity of P when P_1 can be transformed into the form $D_1+\sqrt{-1}x_1^kD_n$ in a neighborhood of $(0, \sqrt{-1}dx_n)\in\sqrt{-1}T^*\boldsymbol{R}^n$ with a positive odd integer k (cf. [5]). Theorem 2 generalizes our previous result (Corollary of [4]) which corresponds to the case $k=1$. To prove Theorem 2, we use different methods from those sketched in [4]; Schapira's theory of positivity (cf. [6]) enables us to reduce the problem of analytic hypo-ellipticity to that of propagation of micro-analyticity of solutions of such equations as treated in § 1.

§ 1. Propagation of regularity. Set $X=\boldsymbol{C}^n\ni z=(z_1, \cdots, z_n)$ and $M=\boldsymbol{R}^n\ni x=(x_1, \cdots, x_n)$. We denote by $T^*X=\{(z,\zeta)\in\boldsymbol{C}^n\times\boldsymbol{C}^n\}$ the cotangent bundle of X, by $T_M^*X=\{(x,\sqrt{-1}\eta)\,;\,x\in\boldsymbol{R}^n,\ \eta\in\boldsymbol{R}^n\}$ the conormal bundle of M in X, and by $\mathcal{C}_M$ the sheaf on T_M^*X of microfunctions. For holomorphic functions f and g defined on an open subset of T^*X, we set

$$H_f=\sum_{j=1}^n\left(\frac{\partial f}{\partial\zeta_j}\frac{\partial}{\partial z_j}-\frac{\partial f}{\partial z_j}\frac{\partial}{\partial\zeta_j}\right)$$

and $\{f,g\}=H_fg$, and denote by f^c the complex conjugate of f with respect to T_M^*X; i.e., f^c is the unique holomorphic function such that $f^c=\bar{f}$ holds on T_M^*X. We denote by σ the principal symbol of a microdifferential operator of finite order, and by σ_j the symbol of order j when the operator is of order at most j.

Let P_1 and P_2 be microdifferential operators of order l_1 and l_2 respectively defined in a neighborhood of $p\in T_M^*X-M$. Set $l=l_1+l_2$ and let $Q=(Q_{ij})$ be an $m\times m$ matrix of microdifferential operators of order at most $l-1$. We assume

(A.1) $\sigma(P_1)(p)=\sigma(P_2)(p)=0$,

(A.2) $\{\sigma(P_1), \sigma(P_2)\}(p)\neq 0$.

Let $\lambda_1, \cdots, \lambda_m$ be the eigenvalues of the matrix

$$(\sigma_{l-1}(Q_{ij})(p)/\{\sigma(P_1), \sigma(P_2)\}(p)).$$

Then we also assume

(A.3) $\lambda_j \notin \{0, 1, 2, \cdots\}$ for $j=1, \cdots, m$.

Set $V_j=\{(z,\zeta)\in T^*X\,;\,\sigma(P_j)(z,\zeta)=0\}$ and

$$V_j^c=\{(z,\zeta)\in T^*X\,;\,\sigma(P_j)^c(z,\zeta)=0\} \qquad \text{for } j=1,2.$$

In this section we assume

(R) $V_1=V_1^c$.

Then $V_1^R=V_1\cap T_M^*X$ is a 1-codimensional real analytic submanifold of T_M^*X. There is a real valued real analytic function f defined in a neighborhood (in T_M^*X) of p such that $V_1^R=\{f=0\}$, that df and $\eta_1 dx_1+\cdots+\eta_n dx_n$ are linearly independent, and that f is homogeneous with respect to η. Then each maximal integral curve of the real vector field $\sqrt{-1}H_f$ on V_1^R is called a bicharacteristic of V_1^R.

Now let $b_1(p)$ be the bicharacteristic of V_1^R through p. Then we have the following

Theorem 1. *Set $P=P_1P_2I_m+Q$, where P_1, P_2, Q satisfy the conditions* (A.1)–(A.3) *and* (R). *Let u be a column vector of m microfunctions defined in a neighborhood of p such that $Pu=0$ and that u vanishes on $b_1(p)-\{p\}$. Then u vanishes on $b_1(p)$.*

Now we give a sketch of the proof. Firstly, we may assume that $P_1=x_1$ and that $p=(0, \sqrt{-1}dx_n)$ by a real quantized contact transformation. Set $N=\{x\in M\,;\,x_1=0\}$ and $Y=\{z\in X\,;\,z_1=0\}$. We define the map $\rho: T_M^*X|_N\to T_N^*Y$ by $\rho(0, x', \sqrt{-1}\eta)=(x', \sqrt{-1}\eta')$, where $x'=(x_2, \cdots, x_n)$ and $\eta'=(\eta_2, \cdots, \eta_n)$. Put $p'=(0, \sqrt{-1}dx_n)\in T_N^*Y$. Then it is sufficient to prove the following proposition in view of Theorem 2.5 of [3]. (We use the notation $D_j=\partial/\partial x_j$.)

Proposition. *Set*

$$P=x_1(D_1I_m-A(x,D'))-B(x',D')\,;$$

here $A=(A_{ij})$ and $B=(B_{ij})$ are $m\times m$ matrices of microdifferential operators defined in a neighborhood of $\rho^{-1}(p')$ such that

(i) *the order of $A_{ij}\leqq 1$, the order of $B_{ij}\leqq 0$.*

(ii) $[x_1, A]=[x_1, B]=[D_1, B]=0$.

(iii) *$\mu_i\notin \boldsymbol{Z}$ and $\mu_i-\mu_j\notin \boldsymbol{Z}-\{0\}$ hold for $1\leqq i, j\leqq m$, or else $\mu_1=\cdots=\mu_m\in\{-1,-2,-3,\cdots\}$ holds, where $\mu_1, \cdots, \mu_m$ are the eigenvalues of $\sigma_0(B)(p')$.*

Under these assumptions, the homomorphism

$$P:(\rho_1\mathcal{C}_M)_{p'}^m\longrightarrow(\rho_1\mathcal{C}_M)_{p'}^m$$

is injective.

To prove this proposition, we use the sheaf $\mathcal{C}_{M+|X}$ and the theory of mild hyperfunctions both developed by Kataoka (cf. [1], [2]).

Example 1. Put $x=(x_1, x_2)\in \boldsymbol{R}^2$ and set

$$P=D_1(D_1-\sqrt{-1}x_1D_2)+a_1(x)D_1+a_2(x)D_2+b(x);$$

here a_1, a_2, b are real analytic functions defined in an open subset U of $\boldsymbol{R}^2$. Assume that $a_2(0, x_2)\notin\{0, -\sqrt{-1}, -2\sqrt{-1}, \cdots\}$ for $(0, x_2)\in U$. Let u be a hyperfunction defined in U such that Pu is real analytic in U and that u is real analytic in $\{x\in U\,;\, x_1\neq 0\}$. Then u is real analytic in U.

§ 2. Analytic hypo-ellipticity. Let P_1, P_2, Q be as in § 1 satisfying (A.1)–(A.3). In this section, we assume

(H) $V_1\cap V_1^c$ is non-singular, and the pull back of $d\zeta_1\wedge dz_1+\cdots+d\zeta_n\wedge dz_n$ to $V_1\cap V_1^c$ is non-degenerate on a neighborhood of p; there are a positive odd integer k and a complex number a such that

$$(H_{\sigma(P_1)})^j\sigma(P_1)^c(p)=0 \qquad \text{for } 0\leqq j\leqq k-1,$$
$$a^{k-1}(H_{\sigma(P_1)})^k\sigma(P_1)^c(p)<0,$$
$$(k-1)(d(a\sigma(P_1))+d(a\sigma(P_1))^c)(p)=0.$$

Theorem 2. *Under the assumptions* (A.1)–(A.3) *and* (H), *the homomorphism*

$$P_1P_2I_m+Q:(\mathcal{C}_M)_p^m\longrightarrow(\mathcal{C}_M)_p^m$$

is injective.

We give a sketch of the proof. First, note that P_1 is equivalent to the operator $D_1+\sqrt{-1}x_1^kD_n$ with $p=(0, \sqrt{-1}dx_n)$ by a real quantized contact transformation (cf. [5]). Then it is easy to see that there is a complex contact transformation φ such that $(T_M^*X, \boldsymbol{C}^\times\varphi(T_M^*X))$ is positive at p in the sense of Schapira [6], $T_M^*X\cap\varphi(T_M^*X)=\{(x, \sqrt{-1}\eta)\in T_M^*X\,;\, x_1=0\}$, and $\zeta_1\circ\varphi=\zeta_1+\sqrt{-1}z_1^k\zeta_n$. Then Theorem 2 follows from Theorem 1 in view of Corollaire 3.4 of [6].

Example 2. Put $x=(x_1, x_2)\in\boldsymbol{R}^2$ and set

$$P=(D_1+\sqrt{-1}x_1^kD_2)(D_1^l-\sqrt{-1}x_1D_2^l)+\sum_{\alpha=(\alpha_1,\alpha_2)\geqq 0,|\alpha|\leqq l}a_\alpha(x)D_1^{\alpha_1}D_2^{\alpha_2};$$

here $a_\alpha(x)$ are real analytic functions defined in an open subset U of $\boldsymbol{R}^2$, and k, l are positive odd integers. Assume that $a_{(0,l)}(0, x_2)\notin\sqrt{-1}\boldsymbol{Z}$ for $(0, x_2)\in U$. Then P is analytic hypo-elliptic in U; i.e., if u is a hyperfunction defined in an open subset U' of U such that Pu is real analytic in U', then u is real analytic in U'.

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