2. An Asymptotic Formula for the Eigenvalues of the Laplacian in a Domain with a Small Hole

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(Communicated by Kôsaku YOSIDA, M. J. A., Jan. 12, 1982)

§1. **Introduction.** This note is a continuation of our previous paper [1]. Let Ω be a bounded domain in $\mathbf{R}^3$ with C^∞ boundary γ and w be a fixed point in Ω. For any sufficiently small $\varepsilon>0$, let B_ε be the ball defined by $B_\varepsilon=\{z\in\Omega\,;\,|z-w|<\varepsilon\}$. Let Ω_ε be the bounded domain defined by $\Omega_\varepsilon=\Omega\backslash\bar{B}_\varepsilon$. Then $\partial\Omega_\varepsilon=\gamma\cup\partial B_\varepsilon$.

Let $0>\mu_1(\varepsilon)\geq\mu_2(\varepsilon)\geq\cdots$ be the eigenvalues of the Laplacian in Ω_ε under the Dirichlet condition on $\partial\Omega_\varepsilon$. Let $0>\mu_1\geq\mu_2\geq\cdots$ be the eigenvalues of the Laplacian in Ω under the Dirichlet condition on γ. We arrange them repeatedly according to their multiplicities.

The aim of this note is to give an asymptotic expression of $\mu_j(\varepsilon)$ as ε tends to zero. We need some notations to state the main result.

Let $G(x,y)$ be the Green function of the Laplacian in Ω satisfying

$$\Delta_x G(x,y)=-\delta(x-y) \qquad x,y\in\Omega,$$
$$G(x,y)|_{x\in\gamma}=0 \qquad y\in\Omega.$$

Then the Robin constant τ $(=\tau(w))$ at w is defined by

$$\tau=\lim_{x\to w}(G(x,w)-(4\pi)^{-1}|x-w|^{-1}).$$

Let $\boldsymbol{G}$ be the Green operator defined by

$$(\boldsymbol{G}f)(x)=\int_\Omega G(x,y)f(y)dy \tag{1.1}$$

for $x\in\Omega$.

We have the following

Theorem 1. *Fix j. Assume that the multiplicity of μ_j is one, then*

$$\begin{aligned}\mu_j(\varepsilon)-\mu_j=&-(\tau+(4\pi\varepsilon)^{-1})^{-1}\varphi_j(w)^2\\&-(\tau+(4\pi\varepsilon)^{-1})^{-2}e_j(w)\varphi_j(w)+O(\varepsilon^{5/2})\end{aligned} \tag{1.2}$$

as ε tends to zero. Here $\varphi_j(x)$ denotes the eigenfunction of the Laplacian under the Dirichlet condition on γ satisfying

$$\int_\Omega \varphi_j(x)^2dx=1.$$

And here

$$e_j(w)=\lim_{x\to w}(G(x,w)\varphi_j(w)+\psi(x)), \tag{1.3}$$

where $\psi\in L^2(\Omega)$ is the unique solution of

$$((\boldsymbol{G}+(1/\mu_j))\psi)(x)=-(1/\mu_j)G(x,w)\varphi_j(w)-(1/\mu_j^2)\varphi_j(w)^2\varphi_j(x) \tag{1.4}$$

and

(1.5) $$\int_{\Omega} \psi(x)\varphi_j(x)dx=0.$$

Remarks. The remainder term in (1.2) is not uniform with respect to j. The above formula (1.2) is a refinement of the formula (1.2) in [1]. See also [3], [2]. From (1.4) it is easily seen that $G(x, w)\varphi_j(w)+\psi(x)$ is continuous with respect to x.

In § 2 we give a rough sketch of our proof of Theorem 1. Details of this paper will appear elsewhere.

§ 2. Sketch of proof of Theorem 1.

Step 1. Let τ be as before. Put $\varepsilon^*=(\tau+(4\pi\varepsilon)^{-1})^{-1}$. Put $\beta_\varepsilon=\{x \in \Omega\,;\, G(x, w)>\varepsilon^{*-1}\}$ and $w_\varepsilon=\Omega\backslash\bar{\beta}_\varepsilon$. Then it is easy to see that there exists a constant C (>0) independent of ε satisfying

(2.1) $$\Omega_{\varepsilon+C\varepsilon^3}\subset\omega_\varepsilon\subset\Omega_{\varepsilon-C\varepsilon^3}.$$

Let $0>\tilde{\mu}_1(\varepsilon)\geq\cdots\geq\tilde{\mu}_j(\varepsilon)\cdots$ be the eigenvalues of the Laplacian in ω_ε under the Dirichlet condition on $\partial\omega_\varepsilon$. If we prove

(2.2) $$\tilde{\mu}_j(\varepsilon)-\mu_j=-\varepsilon^*\varphi_j(w)^2-(\varepsilon^*)^2e_j(w)\varphi_j(w)+O(\varepsilon^{5/2})$$

when the multiplicity of μ_j is one, then we get (1.2) because of (2.1). Thus we have only to prove (2.2) to obtain Theorem 1.

Step 2. Put $G^{(1)}(x, y)=G(x, y)$ and

$$G^{(k)}(x, y)=\int_{\Omega} G^{(k-1)}(x, z)G(z, y)dz$$

inductively. We define the symbol $\langle\nabla_w, \nabla_w\rangle$ by the following:

$$\langle\nabla_w a(x, w), \nabla_w b(y, w)\rangle=\sum_{i=1}^{3}\frac{\partial a}{\partial w_i}(x, w)\frac{\partial b}{\partial w_i}(y, w).$$

We now introduce the integral kernel $h_\varepsilon(x, y)$ by the following:

(2.3) $$\begin{aligned} h_\varepsilon(x,y)=&G^{(3)}(x, y)-\varepsilon^*\sum_{k=1}^{3}G^{(k)}(x, w)G^{(4-k)}(y, w)\\ &-4\pi\varepsilon^3\sum_{k=1}^{3}\langle\nabla_w G^{(k)}(x, w), \nabla_w G^{(4-k)}(y, w)\rangle\xi_\varepsilon(x)\xi_\varepsilon(y)\\ &+(\varepsilon^*)^2G^{(2)}(w, w)\sum_{k=1}^{2}G^{(k)}(x, w)G^{(3-k)}(y, w)\\ &+(\varepsilon^*)^2G^{(3)}(w, w)G(x, w)G(y, w),\end{aligned}$$

where $\xi_\varepsilon(x)\in C^\infty(\Omega)$ is a function satisfying $|\xi_\varepsilon(x)|\leq 1$, $\xi_\varepsilon(x)=0$ for $x\in\beta_{\varepsilon/2}$, $\xi_\varepsilon(x)=1$ for $x\in\omega_{(3/4)\varepsilon}$.

Let $\boldsymbol{H}_\varepsilon$ be the operator given by

(2.4) $$(\boldsymbol{H}_\varepsilon f)(x)=\int_{\omega_\varepsilon} h_\varepsilon(x, y)f(y)dy$$

for $x\in\omega_\varepsilon$. And let $\boldsymbol{G}_\varepsilon$ be the operator given by

(2.5) $$(\boldsymbol{G}_\varepsilon g)(x)=\int_{\omega_\varepsilon} G_\varepsilon(x, y)g(y)dy$$

for $x\in\omega_\varepsilon$, where $G_\varepsilon(x, y)$ be the Green kernel of the Laplacian in ω_ε under the Dirichlet condition on $\partial\omega_\varepsilon$.

We have the following

Proposition 1. *There exists a constant $C>0$ independent of ε*

such that

(2.6) $$\|G_\varepsilon^3 - H_\varepsilon\|_{L^2(\omega_\varepsilon)} \leq C\varepsilon^{5/2}$$

holds, where $\|T\|_{L^2(\omega_\varepsilon)}$ *denotes the operator norm of* T.

We need hard and laborious calculations including L^p-spaces to get (2.6).

Step 3. Let $\tilde{H}_\varepsilon$ be the operator given by

(2.7) $$(\tilde{H}_\varepsilon g)(x) = \int_\Omega h_\varepsilon(x, y) g(y) dy$$

for $x \in \Omega$.

We here construct an approximate eigenvalue and an approximate eigenfunction of $\tilde{H}_\varepsilon$. Put $\lambda = -1/\mu_j$ and $\lambda_1 = -3\lambda^4 \varphi_j(w)^2$. We consider the following equations:

(2.8) $$(G - \lambda)\Phi(x) = \lambda\varphi_j(w)(G(x, w) - \lambda\varphi_j(w)\varphi_j(x)),$$

(2.9) $$\int_\Omega \Phi(x)\varphi_j(x) dx = 0.$$

Since $G - \lambda$ is the operator of Fredholm type, the unique solution Φ in $L^2(\Omega)$ of (2.8), (2.9) exists. Put

$$\lambda_2 = \lambda^2 \varphi_j(w)^2 (2\lambda G^{(2)}(w, w) + G^{(3)}(w, w)) - \lambda\varphi_j(w) \sum_{k=0}^{2} \lambda^k (G^{3-k}\Phi)(w).$$

Then consider the equations:

$$\begin{aligned}(G^3 - \lambda^3)\Psi(x) = {} & \lambda_2 \varphi_j(x) + \lambda_1 \Phi(x) + \sum_{k=0}^{2} G^{(k+1)}(x, w)(G^{3-k}\Phi)(w) \\ & - \left(\sum_{k=1}^{2} \lambda^{3-k} G^{(k)}(x, w) G^{(2)}(w, w) \right. \\ & \left. + \lambda G(x, w) G^{(3)}(w, w) \right) \varphi_j(w)\end{aligned} \tag{2.10}$$

and

(2.11) $$\int_\Omega \Psi(x)\varphi_j(x) dx = 0.$$

We see that (2.10), (2.11) have the unique solution Ψ in $L^2(\Omega)$. Now put $\tilde{\lambda}(\varepsilon) = \lambda^3 + \varepsilon^* \lambda_1 + (\varepsilon^*)^2 \lambda_2$ and $\tilde{\varphi}(\varepsilon) = \varphi_j + \varepsilon^* \Phi + (\varepsilon^*)^2 \Psi$. We have the following

Proposition 2. *There exists a constant* $C > 0$ *independent of* ε *such that*

(2.12) $$\|(\tilde{H}_\varepsilon - \tilde{\lambda}(\varepsilon))\tilde{\varphi}(\varepsilon)\|_{L^2(\Omega)} \leq C\varepsilon^{5/2}$$

holds.

Let $\chi(\varepsilon)$ be the characteristic function of ω_ε. Then we can prove

(2.13) $$\|\chi(\varepsilon)\tilde{H}_\varepsilon \tilde{\varphi}(\varepsilon) - H_\varepsilon(\chi(\varepsilon)\tilde{\varphi}(\varepsilon))\|_{L^2(\omega_\varepsilon)} \leq C\varepsilon^{5/2}$$

for a constant C independent of ε.

By (2.6), (2.12) and (2.13) we get the following

Proposition 3. *The inequality*

(2.14) $$\|(G_\varepsilon^3 - \tilde{\lambda}(\varepsilon))\chi(\varepsilon)\tilde{\varphi}(\varepsilon)\|_{L^2(\omega_\varepsilon)} \leq C\varepsilon^{5/2}$$

holds.

As a consequence of (2.14), we conclude the following

Proposition 4. *There exists at least one eigenvalue* $\lambda^*(\varepsilon)$ *of* $\boldsymbol{G}_\varepsilon$ *satisfying*

$$\lambda^*(\varepsilon)^3 = \lambda^3 + \varepsilon^* \lambda_1 + (\varepsilon^*)^2 \lambda_2 + O(\varepsilon^{5/2}) \tag{2.15}$$

as ε *tends to zero.*

By the result of Rauch-Taylor [4], we see that there exists exactly one eigenvalue $\lambda^*(\varepsilon)$ of $\boldsymbol{G}_\varepsilon$ satisfying (2.15). Therefore, (2.2) is proved.

References

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