12. On Hilbert Modular Forms. II

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Introduction. In his paper [5], J. Igusa gave a minimal set of generators over $\boldsymbol{Z}$ of the graded ring of Siegel modular forms of genus two whose Fourier coefficients lie in $\boldsymbol{Z}$. Also, some problems on the finite generation of an algebra of modular forms were discussed by W. L. Baily, Jr. in his recent paper [1]. The author studied the structure of graded $\boldsymbol{Z}[1/2]$-algebra of symmetric Hilbert modular forms for $\boldsymbol{Q}(\sqrt{5})$ in his first paper [6]. The purpose of this second paper is to describe the minimal sets of generators over $\boldsymbol{Z}$ of the graded rings of symmetric Hilbert modular forms with integral Fourier coefficients for real quadratic fields $\boldsymbol{Q}(\sqrt{2})$ and $\boldsymbol{Q}(\sqrt{5})$. The detailed results with their complete proofs will appear elsewhere.

§ 1. Hilbert modular forms for real quadratic fields. Let $\boldsymbol{K}$ be a real quadratic field and let $\mathfrak{o}_K$ denote the ring of integers in $\boldsymbol{K}$. Let $\mathfrak{H}$ denote the upper-half plane and we put $\mathfrak{H}^2=\mathfrak{H}\times\mathfrak{H}$. We denote by $A_C(\Gamma_K)_k$ the set of symmetric Hilbert modular forms of weight k for $\boldsymbol{K}$, where $\Gamma_K=SL(2,\mathfrak{o}_K)$ is the Hilbert modular group. Let $\mathfrak{d}_K$ denote the different of $\boldsymbol{K}$. Then any element in $A_C(\Gamma_K)_k$ has the following Fourier expansion:

$$f(\tau)=\sum_{\nu\in\mathfrak{d}_K^{-1}} a_f(\nu)\exp[2\pi i tr(\nu\tau)],$$

where the sum extends over the elements ν in $\mathfrak{d}_K^{-1}$ which are totally positive or 0. For any subring R of $\boldsymbol{C}$, define in $A_C(\Gamma_K)_k$ the subset

$$A_R(\Gamma_K)_k=\{f(\tau)\in A_C(\Gamma_K)_k \mid a_f(\nu)\in R \text{ for all } \nu\in\mathfrak{d}_K^{-1},\ \nu\gg 0 \text{ or } 0\}.$$

$A_R(\Gamma_K)_k$ is an R-module, and if we put $A_R(\Gamma_K)=\oplus_{k\geqq 0}A_R(\Gamma_K)_k$, then $A_R(\Gamma_K)$ is a graded R-algebra. Next we shall introduce the Eisenstein series $G_k(\tau)$ of weight k for Γ_K. Let $\sim$ denote an equivalence relation in $\mathfrak{o}_K\times\mathfrak{o}_K$ defined as follows:

$$(\alpha,\beta)\sim(\alpha',\beta') \text{ if } \alpha'=\varepsilon'\alpha,\ \beta'=\varepsilon'\beta \text{ for some unit } \varepsilon' \text{ in } K.$$

For any even integer $k\geqq 2$, we define a series $G'_k(\tau)$ on $\mathfrak{H}^2$ as:

$$G'_k(\tau)=\sum_{(\lambda,\mu)\in\mathfrak{o}_K\times\mathfrak{o}_K/\sim} N(\lambda\tau+\mu)^{-k},\qquad \tau\in\mathfrak{H}^2.$$

where the summation runs through a set of representatives (λ,μ) $\neq(0,0)$. It is well known that the series is absolutely convergent and represents a symmetric Hilbert modular form of weight k for $\boldsymbol{K}$.

We normalize $G'_k(\tau)$ as:

$$G_k(\tau)=\zeta_K(k)^{-1}\cdot G'_k(\tau),$$

where $\zeta_K(s)$ is the Dedekind zeta function for K. The function $G_k(\tau)$ is called the *normalized Eisenstein series of weight* k *for* Γ_K and it has the following Fourier expansion:

$$G_k(\tau)=1+\kappa_k \sum_{\substack{\nu\in\mathfrak{d}_K^{-1}\\ \nu\gg 0}} \sigma_{k-1}(\nu)\exp[2\pi i tr(\nu\tau)],$$

$$\kappa_k=\zeta_K(k)^{-1}\cdot(2\pi i)^{2k}\cdot[(k-1)!]^{-2}\cdot d_K^{1/2-k},$$

$$\sigma_{k-1}(\nu)=\sum_{(\nu)\mathfrak{d}_K\subset\mathfrak{b}} |N(\mathfrak{b})|^{k-1},$$

where d_K is the discriminant of K. From Hecke's result it follows that

$$\zeta_K(k)=\pi^{2k}\cdot d_K^{1/2}\cdot(\text{rational number}),$$

so we see that $G_k(\tau)\in A_Q(\Gamma_K)_k$. Now we denote with $A_C(SL(2,Z))_m$ the complex vector space of all elliptic modular forms of weight m. (We define $A_R(SL(2,Z))_m$, $A_R(SL(2,Z))$ in similar way). For any function $f(\tau)$ on $\mathfrak{H}^2$, we define a function $D(f)(z)$ on $\mathfrak{H}$ by $D(f)(z)=f((z,z))$. By definition, if $f(\tau)$ is a function in $A_C(\Gamma_K)_k$, then $f((z,z))\in A_C(SL(2,Z))_{2k}$. Furthermore, if we assume that the function $f(\tau)$ has the Fourier expansion of the form

$$f(\tau)=\sum a_f(\nu)\exp[2\pi i tr(\nu\tau)],$$

then $D(f)(z)$ has the following Fourier expansion:

$$D(f)(z)=\sum_{n=0}^{\infty} c_f(n)\exp(2\pi i n z),\qquad c_f(n)=\sum_{tr(\nu)=n} a_f(\nu).$$

From this result, we see also that, if $f\in A_R(\Gamma_K)_k$, then $D(f)\in A_R(SL(2,Z))_{2k}$.

§2. **Main results.** In this section, we shall state the main results. Namely, we give the minimal sets of generators over Z of $A_Z(\Gamma_K)$ for $K=Q(\sqrt{2})$ and $Q(\sqrt{5})$. Let $E_k(z)$ be the normalized Eisenstein series of weight k for $SL(2,Z)$ and $\Delta(z)$ be a cusp form of weight 12 defined by $\Delta(z)=2^{-6}\cdot 3^{-3}(E_4^3(z)-E_6^2(z))$. It is well known that $\Delta(z)$ has the expression: $\Delta(z)=q\prod_{n=1}^{\infty}(1-q^n)^{24}$, $q=\exp(2\pi i z)$.

(1) *The case* $K=Q(\sqrt{2})$. In this case, we have $d_K=8$, $\mathfrak{d}_K=(2\sqrt{2})$. The first few examples of Fourier expansions of the Eisenstein series $G_k(\tau)$ are given as follows:

$$G_2(\tau)=1+2^4\cdot 3\sum\sigma_1(\nu)\exp[2\pi i tr(\nu\tau)],$$

$$G_4(\tau)=1+2^5\cdot 3\cdot 5\cdot 11^{-1}\sum\sigma_3(\nu)\exp[2\pi i tr(\nu\tau)],$$

$$G_6(\tau)=1+2^4\cdot 3^2\cdot 7\cdot 19^{-2}\sum\sigma_5(\nu)\exp[2\pi i tr(\nu\tau)],$$

(e.g., cf. [2], p. 321).

Proposition 1.1. *Under the above notations, we have*

$$D(G_2)=E_4,\quad D(G_4)=E_8=E_4^2,\quad D(G_2^3-G_6)=2^7\cdot 3^3\cdot 5\cdot 13\cdot 19^{-2}\Delta.$$

Now we put

$$V_2=G_2,\qquad V_4=2^{-6}\cdot 3^{-2}\cdot 11(G_2^2-G_4),$$

$$V_6=2^{-8}\cdot 3^{-3}\cdot 13^{-1}\cdot 1471G_2^3-2^{-8}\cdot 3^{-1}\cdot 5^{-1}\cdot 13^{-1}\cdot 11\cdot 67G_2G_4 -2^{-7}\cdot 3^{-3}\cdot 5^{-1}\cdot 13^{-1}\cdot 19^2G_6.$$

Proposition 1.2. *The above-defined three modular forms have integral Fourier coefficients,* i.e., $V_k \in A_Z(\Gamma_K)_k$ *for* $k=2, 4, 6$. *Furthermore,* $D(V_2)=E_4$, $D(V_4)=0$, $D(V_6)=\Delta$.

We note that the forms V_4 and V_6 have the expressions by the theta functions.

Theorem 1. *The elements* V_2, V_4, V_6 *form a minimal set of generators of* $A_Z(\Gamma_K)$ *over* Z.

(2) *The case* $K=Q(\sqrt{5})$. In this case, $d_K=5$ and $\mathfrak{d}_K=(\sqrt{5})$. The Fourier expansions of G_k ($k=2, 4, 6, 10$) are given as follows:

$$G_2(\tau)=1+2^3\cdot 3\cdot 5\sum\sigma_1(\nu)\exp[2\pi i tr(\nu\tau)],$$
$$G_4(\tau)=1+2^4\cdot 3\cdot 5\sum\sigma_3(\nu)\exp[2\pi i tr(\nu\tau)],$$
$$G_6(\tau)=1+2^3\cdot 3^2\cdot 5\cdot 7\cdot 67^{-1}\sum\sigma_5(\nu)\exp[2\pi i tr(\nu\tau)],$$
$$G_{10}(\tau)=1+2^3\cdot 3\cdot 5^2\cdot 11\cdot 412751^{-1}\sum\sigma_9(\nu)\exp[2\pi i tr(\nu\tau)].$$

Proposition 2.1. *Under the above definitions, we have*

$$D(G_2)=E_4, \qquad D(G_2^3-G_6)=2^6\cdot 3^3\cdot 5^2\cdot 67^{-1}\Delta.$$

We put

$$W_2=G_2, \qquad W_6=2^{-5}\cdot 3^{-3}\cdot 5^{-2}\cdot 67(G_2^3-G_6),$$
$$W_{10}=2^{-10}\cdot 3^{-5}\cdot 5^{-5}\cdot 7^{-1}(412751G_{10}-5\cdot 67\cdot 2293G_2^2G_6+2^2\cdot 3\cdot 7\cdot 4231G_2^5),$$
$$W_{12}=2^{-2}(W_6^2-W_2W_{10}).$$

Proposition 2.2. *The four modular forms* W_2, W_6, W_{10}, W_{12} *all have integral Fourier coefficients. Furthermore* $D(W_2)=E_4$, $D(W_6)=2\Delta$, $D(W_{10})=0$, $D(W_{12})=\Delta^2$.

We should remark that the modular form W_{10} coincides with a cusp form Θ^2 defined by Gundlach [3] up to constant.

Theorem 2. *The elements* W_2, W_6, W_{10}, W_{12} *form a minimal set of generators of* $A_Z(\Gamma_K)$ *over* Z.

The result of Theorem 2 is a consequence of the fact that $D(A_Z(\Gamma_K))=Z[E_4, 2\Delta, \Delta^2]$.

References

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