118. *Energy Inequality for Non Strictly Hyperbolic Operators in the Gevrey Class*

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1. Introduction. In this article, we shall establish the energy inequality for non strictly hyperbolic operators of order m whose coefficients are in the Gevrey class with respect to the space variables and once or twice continuously differentiable with respect to the time variable.

More precisely, we consider the following Cauchy problem,

$$(1.1)\quad \begin{cases} P(y, D_0, D)u = D_0^m u + \sum_{\substack{|\alpha|+j\le m \\ j\le m-1}} a_{\alpha,j}(y) D^\alpha D_0^j u = f & \text{in } G \\ D_0^j u(t_0, x) = u_j(x)\ j=0,1,\cdots,m-1 & \text{in } G_{t_0} = G \cap \{x_0 = t_0\}. \end{cases}$$

where $y=(x_0, x)=(x_0, x_1, \cdots, x_d)$,

$$D_0 = \frac{1}{i}\frac{\partial}{\partial x_0}, \qquad D = \left(\frac{1}{i}\frac{\partial}{\partial x_1}, \cdots, \frac{1}{i}\frac{\partial}{\partial x_d}\right),$$

G is a lens of spatial type in R_y^{d+1}.

Let us denote by $P_m(y, \xi_0, \xi)$ the principal symbol of $P(y, D_0, D)$,

$$(1.2)\qquad P_m(y, \xi_0, \xi) = \xi_0^m + \sum_{\substack{|\alpha|+j=m \\ j\le m-1}} a_{\alpha,j}(y)\xi^\alpha \xi_0^j$$

and assume that P is hyperbolic, that is all the ξ_0-roots of $P_m(y, \xi_0, \xi) = 0$ are real and its multiplicity is at most r $(1 \le r \le m)$, for any $y \in G$, any $\xi \in R^d \backslash \{0\}$. For an open set G in $R_{x_0} \times R_x^d$, we denote by $\gamma^{K,(s)}(G)$, where $K=0, 1, \cdots$, and $1 < s < \infty$, the set of functions $f(y)$ such that for any compact set $M \subset G$, there exist constants C, A satisfying the inequalities

$$(1.3)\qquad \sup_{y\in M} |D_0^j D^\alpha f(y)| \le CA^{|\alpha|}(|\alpha|!)^s$$

for all $\alpha \in N^d$, $0 \le j \le K$. We also denote $\gamma^{(s)}(\Omega)$, where Ω is an open set in R_x^d, the set of all functions $g(x)$ such that for any compact set $M \subset \Omega$, the following estimates are valid with some constants C, A for all $\alpha \in N^d$

$$(1.4)\qquad \sup_{x\in M} |D^\alpha g(x)| \le CA^{|\alpha|}(|\alpha|!)^s.$$

Recently, M. D. Bronshtein [2] has proved, by constructing the parametrix of which the remainder is a bounded operator in some Hilbert space connected with the Gevrey class, that the problem (1.1) has a unique solution in $\tilde{G}$ $(\subset G)$ for any initial data $u_j(x) \in \gamma^{(s)}(G_{t_0})$, if $a_{\alpha,j}(y) \in \gamma^{K,(s)}(G)$, $K > 3(m+d+2)$, $1 \le s \le r/(r-1)$. And if $1 \le s < r/(r-1)$, one can take $\tilde{G}$ so that it depends only on P.

In this paper, we shall establish the energy inequality for non strictly hyperbolic operators whose coefficients belong to $\gamma^{K,(s)}(G)$, where $1<s\leq(r-(1/2))/(r-1)$, $K=1$ or $1<s\leq r/(r-1)$, $K\geq 2$. From the energy inequalities obtained in the present paper, it follows that the problem (1.1) has a unique solution in $\tilde{G}$ ($\subset G$) for any initial data $u_j(x) \in \gamma^{(s)}(G_{t_0})$ if $a_{\alpha,j}(y)\in\gamma^{2,(s)}(G)$ and $1<s\leq r/(r-1)$. The same assertion is valid for any data belonging to $\gamma^{(s)}(G_{t_0})$ if $a_{\alpha,j}(y)\in\gamma^{1,(s)}(G)$ and $1<s\leq(r-(1/2))/(r-1)$. If $1<s<r/(r-1)$ in the first case and $1<s<(r-(1/2))/(r-1)$ in the second case, we can take $\tilde{G}=G$. Here, for the simplicity, we have assumed that f in (1.1) is zero.

In general, the regularity with respect to x_0 that we have imposed on the coefficients of P is the best possible when we measure it by C^K class. For the second order non strictly hyperbolic operators, more complete results about the relation between the order of the class of admissible data and the regularity of the time variable are obtained in [3], [8].

2. Statement of results. From now on, we assume that $a_{\alpha,j}(y)$ belongs to $\gamma^{K,(s)}(J\times R^d_x)$, $J=(-T,T)$ and $a_{\alpha,j}(y)$ is independent of x if $x\notin M$, with some compact neighborhood M of the origin in R^d_x. This is easily made by replacing $a_{\alpha,j}(y)$ by $a_{\alpha,j}(x_0,\nu(x_1),\cdots,\nu(x_d))$ with suitable function $\nu(x)$. The resulting operator is hyperbolic and the multiplicity of the characteristic roots is at most r for $y\in J\times R^d_x$, $\xi\in R^d_\xi\backslash\{0\}$, and this coincides with the original operator in $J\times\{x\in R^d_x;\ |x|<R\}$.

For $u(x)\in\gamma^{(s)}(R^d_x)$ with compact support, introduce the norm defined as follows

$$g^p_N(u;\rho)=\sum_{n\geq p}\|\langle D\rangle^{\delta n}u\|^2\rho^{n+N}/(n+N)! \tag{2.1}$$

where $2\delta s=1$ and $p,N\in \mathbf{N}$, $\rho\geq 0$, and $\langle D\rangle^{\delta n}$ denotes the pseudo-differential operator with symbol $\langle\xi\rangle^{\delta n}$, with $\langle\xi\rangle^2=1+\sum_{j=1}^d\xi_j^2$, $\|u\|$ is the L^2-norm in R^d_x.

Let us formulate the main assertions of the present paper.

Theorem 2.1. *Fix the interval* $\tilde{J}=[T_0,T_1]\subset J=(-T,T)$ *arbitrarily and assume that the coefficients are in* $\gamma^{K,(s)}(J\times R^d_x)$. *If* $1<s<(r-(1/2))/(r-1)$, $K=1$ *or* $1<s<r/(r-1)$, $K\geq 2$, *there exist positive constants* C, $c=c(\tilde{J},P)$ *and an integer* L *which do not depend on* γ,N *such that*

$$\begin{aligned}\gamma^{-2r+1}&\sum_{j=0}^{m+K-r-1}g_N^{2m-2r+1}(\langle D\rangle^{2\delta r-4\delta K+3\delta}D_0^ju(\cdot,t);\gamma^{-1})\\ &\leq C\gamma^L\sum_{j=0}^{m+K-1}g_N^{2m-2r+1}(\langle D\rangle^{m-1+\delta}D_0^ju(\cdot,t_0);3\gamma^{-1})\\ &\quad+C\gamma^L\int_{t_0}^{t_0+2\gamma^{-1}}\sum_{i=0}^{K}g_N^{2m}(\langle D\rangle^{-2\delta i}D_0^iPu(\cdot,x_0);t_0+3\gamma^{-1}-x_0)dx_0\end{aligned}$$

$$+C\gamma^{L}\int_{t-\gamma^{-1}}^{t}\sum_{i=0}^{K} g_N^{2m}(\langle D\rangle^{-2\delta i}D_0^iPu(\cdot,x_0);t+\gamma^{-1}-x_0)dx_0$$
$$+C\gamma^{L+1}\int_{t_0}^{t-\gamma^{-1}}d\tau\int_{\tau}^{\tau+2\gamma^{-1}}\sum_{i=0}^{K} g_N^{2m}(\langle D\rangle^{-2\delta i}D_0^iPu(\cdot\,;x_0);\tau+3\gamma^{-1}-x_0)dx_0,$$

for $\gamma\geq\gamma_0$, $N\geq N(\gamma)$, $T_0+8\gamma^{-1}\leq t_0\leq t\leq\min(t_0+c,\ T_1-8\gamma^{-1})$.

Remark 2.1. Since c does not depend on γ, N this theorem assures the existence of the solution in $(T_0,T_1)\times R^d=G$, for any Cauchy data $u_j(x)$ belonging to $\gamma^{(s)}(G_{t_0})$, $T_0<t_0<T_1$.

Theorem 2.2. *Under the same hypotheses those of Theorem* 2.1, *if* $s=(r-(1/2))/(r-1)$ *with* $K=1$ *or* $s=r/(r-1)$ *with* $K\geq 2$, *there exists positive constant* C *such that*

$$\sum_{j=0}^{m+K-r-1} g_N^{2m-2r+1}(\langle D\rangle^{2\delta r-4\delta K+3\delta}D_0^ju(\cdot,t);\gamma(\theta+t_0-t))$$
$$\leq C\sum_{j=0}^{m+K-1} g_N^{2m-2r+1}(\langle D\rangle^{m-1+\delta}D_0^ju(\cdot,t_0);\gamma\theta)$$
$$+C\sum_{i=0}^{K}\int_{t_0}^{t} g_N^{2m}(\langle D\rangle^{-4\delta i}D_0^iPu(\cdot,x_0);\gamma(\theta+t_0-x_0))dx_0,$$

for $\gamma\geq\gamma_0$, $N\geq N(\gamma)$, $T_0+8\gamma^{-1}\leq t_0\leq t\leq\min(t_0+\theta,\ T_1-8\gamma^{-1})$, $0\leq\theta\leq 4\gamma^{-1}$.

The proof is fairly long and is based on the estimates of the regularities of the roots of hyperbolic polynomials in [1] and on the method of proving the sharp Gårding inequality in [4].

A forthcoming paper will give the detailed proof.

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