117. Product Formula for Nonlinear Semigroups in Hilbert Spaces

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1. Introduction. Let H be a real Hilbert space. Let A and B be maximal monotone (multi-valued) operators in H such that $A+B$ is also maximal monotone in H. (We refer to the work of Brezis [2] for basic results concerning maximal monotone operators.) Let $\{S_A(t); t\geqq 0\}$, $\{S_B(t); t\geqq 0\}$ and $\{S_{A+B}(t); t\geqq 0\}$ be the contractive semigroups in H generated by $-A$, $-B$ and $-(A+B)$, respectively. The purpose of this paper is to show the following result.

Theorem. *If there exists a closed convex set $C\subset\overline{D(A)\cap D(B)}$ such that $(I+\lambda A)^{-1}(C)\subset C$ and $(I+\lambda B)^{-1}(C)\subset C$ for $\lambda>0$, then*

(1.1) $$S_{A+B}(t)x=\lim_{n\to\infty}(S_A(t/n)S_B(t/n))^n x$$

for each $x\in C\cap\overline{D(A)\cap D(B)}$ and each $t\geqq 0$ and the convergence is uniform on each finite interval of $[0,\infty)$.

This theorem was proved by Brezis and Pazy in [3] with the extra assumption that A and B are single-valued. Similar results are obtained for some Banach spaces as well and will be treated in the forthcoming paper [5] of the author.

2. Proof of the theorem. (Step 1.) By Proposition 4.5 in [2], $S_A(t)$ and $S_B(t)$ are contractions on C into itself. So we shall prove the convergence

$$\lim_{t\to 0+}(I+\lambda t^{-1}(I-S_A(t)S_B(t)))^{-1}x=(I+\lambda(A+B))^{-1}x$$

for each $x\in C\cap\overline{D(A)\cap D(B)}$ and each $\lambda>0$, from which our assertion is derived through Theorem 4.3 of [2]. To this end, let $\lambda>0$, fix any $x\in C\cap\overline{D(A)\cap D(B)}$ and set $u(t)$ any y_0 to be $(I+\lambda t^{-1}(I-S_A(t)S_B(t))^{-1}x$ and $(I+\lambda(A+B))^{-1}x$, respectively. It can easily be seen that

(2.1) $$\lambda^{-1}(u(t)-x)=t^{-1}(S_A(t)S_B(t)u(t)-u(t)),$$

$u(t)$ are contained in C for all $t>0$ and $u(t)$ is bounded as $t\to 0+$. Since $S_A(t)$ and $S_B(t)$ are contractions from C into itself, the indefinite integrals

$$v(t)=t^{-1}\int_0^t S_B(s)u(t)ds \quad \text{and} \quad w(t)=t^{-1}\int_0^t S_A(s)S_B(t)u(t)ds$$

are contained in C for all $t>0$ and bounded as $t\to 0+$. Therefore, one can choose a null sequence $\{t_n\}$ of positive numbers such that

(2.2) $$u(t_n)\rightharpoonup u_0,\ v(t_n)\rightharpoonup v_0 \quad \text{and} \quad w(t_n)\rightharpoonup w_0$$

as $n\to\infty$, where the symbol $\rightharpoonup$ means the weak convergence and u_0, v_0

and w_0 are elements of C.

(Step 2.) We first show that

$$u_0=v_0=w_0. \tag{2.3}$$

For this purpose, we define a functional $f: H\to[0,\infty)$ by

$$f(y)=\limsup_{n\to\infty}\|y-u(t_n)\|^2 \qquad \text{for } y\in H,$$

which was suggested by the work of Baillon [1]. (See also the work of Opial [6] for basic use of such a functional.) Let $y\in C$. Then, by Minkowski's inequality, we have

$$\begin{aligned}\left(t^{-1}\int_0^t\|S_B(t-s)u(t)-y\|^2\,ds\right)^{1/2} &\leqq\left(t^{-1}\int_0^t\|S_B(t-s)u(t)-S_B(t-s)y\|^2\,ds\right)^{1/2}\\ &\quad+\left(t^{-1}\int_0^t\|S_B(t-s)y-y\|^2\,ds\right)^{1/2}\\ &\leqq\|u(t)-y\|+\left(t^{-1}\int_0^t\|S_B(t-s)y-y\|^2\,ds\right)^{1/2}\end{aligned}$$

and

$$\begin{aligned}\|S_B(t)u(t)-y\|&\leqq\left(t^{-1}\int_0^t\|S_B(t)u(t)-S_B(s)y\|^2\,ds\right)^{1/2}\\ &\quad+\left(t^{-1}\int_0^t\|S_B(s)y-y\|^2\,ds\right)^{1/2}\\ &\leqq\left(t^{-1}\int_0^t\|S_B(t-s)u(t)-y\|^2\,ds\right)^{1/2}\\ &\quad+\left(t^{-1}\int_0^t\|S_B(s)y-y\|^2\,ds\right)^{1/2}.\end{aligned}$$

Let $t=t_n$ and let n tend to the infinity. Then, we have

$$\begin{aligned}&\limsup_{n\to\infty}\|S_B(t_n)u(t_n)-y\|^2\\ &\quad\leqq\limsup_{n\to\infty} t_n^{-1}\int_0^{t_n}\|S_B(t_n-s)u(t_n)-y\|^2\,ds\\ &\quad\leqq f(y),\end{aligned}$$

for $y\in C$. Similarly, we have

$$\begin{aligned}\left(t^{-1}\int_0^t\|S_A(t-s)S_B(t)u(t)-y\|^2\,ds\right)^{1/2} &\leqq\|S_B(t)u(t)-y\|+\left(t^{-1}\int_0^t\|S_A(t-s)y-y\|^2\,ds\right)^{1/2}\end{aligned}$$

and

$$\begin{aligned}\|S_A(t)S_B(t)u(t)-y\|&\leqq\left(t^{-1}\int_0^t\|S_A(t-s)S_B(t)u(t)-y\|^2\,ds\right)^{1/2}\\ &\quad+\left(t^{-1}\int_0^t\|S_A(s)y-y\|^2\,ds\right)^{1/2}\end{aligned}$$

for $y\in C$, which implies

$$\begin{aligned}&\limsup_{n\to\infty}\|S_A(t_n)S_B(t_n)u(t_n)-y\|^2\\ &\quad\leqq\limsup_{n\to\infty} t_n^{-1}\int_0^{t_n}\|S_A(t_n-s)S_B(t_n)u(t_n)-y\|^2\,ds\\ &\quad\leqq\limsup_{n\to\infty}\|S_B(t_n)u(t_n)-y\|^2\end{aligned}$$

for $y\in C$. But, since $u(t)$ is bounded as $t\to0+$, (2.1) implies

$$f(y)=\limsup_{n\to\infty}\|S_A(t_n)S_B(t_n)u(t_n)-y\|^2$$

for $y\in H$. It turns out that

$$(2.4)\qquad f(y)=\limsup_{n\to\infty} t_n^{-1}\int_0^{t_n}\|S_B(t_n-s)u(t_n)-y\|^2\,ds$$
$$=\limsup_{n\to\infty} t_n^{-1}\int_0^{t_n}\|S_A(t_n-s)S_B(t_n)u(t_n)-y\|^2\,ds$$

for all $y\in C$. We now put $C_0=\{y\in C\,;\, f(y)=\inf_{y\in C}f(y)\}$. Since

$$\|u(t)-y\|^2=\|u(t)-u_0\|^2+\|u_0-y\|^2+2\langle u(t)-u_0,\ u_0-y\rangle$$

for $y\in C$, it follows that

$$f(y)\geqq f(u_0)+\|u_0-y\|^2 \qquad \text{for } y\in C.$$

Since $u_0\in C$, it yields $C_0=\{u_0\}$. Similarly, for each $y\in C$,

$$t^{-1}\int_0^t\|S_B(s)u(t)-y\|^2\,ds=t^{-1}\int_0^t\|S_B(s)u(t)-v_0\|^2\,ds$$
$$+\|v_0-y\|^2+2\langle v(t)-v_0,\ v_0-y\rangle,$$

which implies, by (2.4).

$$f(y)\geqq f(v_0)+\|v_0-y\|^2 \qquad \text{for } y\in C.$$

Thus we have $C_0=\{v_0\}$. Similar argument through (2.4) implies $C_0=\{w_0\}$ and hence we obtain (2.3).

(Step 3.) Now let $z_A\in Ay_0$ and $z_B\in By_0$ be such that $y_0+\lambda(z_A+z_B)=x$. By Proposition 3.6 in [2], it follows that

$$\|S_A(t)S_B(t)u(t)-y_0\|^2$$
$$\leqq\|S_B(t)u(t)-y_0\|^2+2\int_0^t\langle -z_A,\ S_A(s)S_B(t)u(t)-y_0\rangle ds$$
$$=\|S_B(t)u(t)-y_0\|^2+2t\langle -z_A,\ w(t)-y_0\rangle.$$

Similarly,

$$\|S_B(t)u(t)-y_0\|^2\leqq\|u(t)-y_0\|^2+2t\langle -z_B,\ v(t)-y_0\rangle.$$

On the other hand, (2.1) implies that

$$\|S_A(t)S_B(t)u(t)-y_0\|^2\geqq\|u(t)-y_0\|^2+2\langle S_A(t)S_B(t)u(t)-u(t),\ u(t)-y_0\rangle$$
$$=\|u(t)-y_0\|^2+2t\lambda^{-1}\langle u(t)-x,\ u(t)-y_0\rangle.$$

Combining these inequalities, we can show that

$$\|u(t)-y_0\|^2\leqq\langle x-y_0,\ u(t)-y_0\rangle+\langle -\lambda z_A,\ w(t)-y_0\rangle$$
$$+\langle -\lambda z_B,\ v(t)-y_0\rangle.$$

Let $t=t_n$ be as in (2.2) and let n tend to the infinity. Since $u(t_n)$, $v(t_n)$ and $w(t_n)$ converge weakly to the same u_0, it follows that

$$\limsup_{n\to\infty}\|u(t_n)-y_0\|^2\leqq\langle x-y_0-\lambda z_A-\lambda z_B,\ u_0-y_0\rangle.$$

Thus, $u(t_n)$ converges strongly to y_0. Q.E.D.

References

[1] J. B. Baillon: Générateurs et semi-groupes dans les espaces de Banach uniformément lisses. J. Funct. Anal., **29**, 199–213 (1978).

[2] H. Brezis: Opérateurs maximaux monotones et semigroupes de contractions dans les espaces de Hilbert. Math. Studies, **6**, North-Holland (1973).

[3] H. Brezis and A. Pazy: Semigroups of nonlinear contractions on convex sets. J. Funct. Anal., **6**, 237–281 (1970).

[4] T. Kato and K. Masuda: Trotter's product formula for nonlinear semigroups generated by the subdifferentials of convex functionals. J. Math. Soc. Japan, **30**, 169–178 (1978).
[5] Y. Kobayashi: Product formula for nonlinear contractive semi-groups in Banach spaces (to appear).
[6] Z. Opial: Lecture Notes on Nonexpansive and Monotone Mappings in Banach Spaces. Lecture Notes **67**-1, Lefschtz Center for Dynamical Systems, Brown University (1967).