101. *Multiplier Algebra of C*-Envelope and the C*-Envelope of a Multiplier Algebra*[*)]

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Abstract. Let A be a commutative Banach *-algebra with $C^*(A)$ as its enveloping C^*-algebra. Denote by $M(B)$ the multiplier algebra of a Banach algebra B. The relations between $M(C^*(A))$ and $C^*(M(A))$ are studied in this note. Let $X=\mathcal{M}(C^*(A))$ and $Y=\mathcal{M}(C^*(M(A)))$ be the maximal ideal spaces of $C^*(A)$ and $C^*(M(A))$ respectively. It is proved that if X is dense in Y then $C^*(M(A))$ can be isometrically embedded as a subalgebra in $M(C^*(A))$. If X is not dense in Y, then it is characterized that there is a homomorphism of $C(Y)$ into $C(\beta(X))$ which is induced from the onto map of $\beta(X)$ to $\tilde{X}$ where $\beta(X)$ is the Stone-Čech compactification of X and $\tilde{X}$ is the weak closure of X in Y.

1. Introduction. Let A be a commutative Banach *-algebra with $C^*(A)$ as its enveloping C^*-algebra. Denote by $M(B)$ the multiplier algebra of some Banach algebra B, that is, a subalgebra of bounded linear operators $\mathcal{L}(B)$ of B which commute with algebra product. It is known that the multiplier algebra of a C^*-algebra is also a C^*-algebra. Thus one will know what relations can be established between $M(C^*(A))$ and $C^*(M(A))$. For example

(i) whether $C^*(M(A))\subset M(C^*(A))$?

(ii) what condition can be $C^*(M(A))\cong M(C^*(A))$?

In general we can not say anything about (i) and (ii). But if the character space $X=\mathcal{M}(C^*(A))$ is dense in the character space $Y=\mathcal{M}(C^*(M(A)))$, then certainly (i) holds. While the condition for (ii) is that A is a dense ideal of $C^*(A)$ containing a bounded approximate identity. If X is not dense in Y, then we find only that there is a homomorphism of $C(Y)$ into $C(\beta(X))$, which is induced from the onto map of $\beta(X)$ to $\tilde{X}$ where $\beta(X)$ is the Stone-Čech compactification of X and $\tilde{X}$ is the weak closure of X in Y.

As an example, if G is a locally compact abelian group with dual group $\hat{G}$, then $\hat{G}$ is homeomorphic to the character space $L^1(G)$ as well as the character space of its enveloping C^*-algebra $C^*(G)$ (cf. Bourbaki [2, p. 113]), but $\hat{G}$ is not dense in the character space Δ of the bounded

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regular measure algebra $M(G)$ except G is compact (see Rudin [9, Corollary 5.3.5]). $M(G)$ is the multiplier algebra of $L^1(G)$, thus $\hat{G}$ is not dense in $Y=\mathcal{M}(C^*(M(G)))$.

2. Multiplier algebras and envelope C^*-algebras. Let B be a commutative C^*-algebra. It is known that the multiplier algebra $M(B)$ of B is also a C^*-algebra containing an identity, and the Gelfand transforms of B is isometrically isomorphic to $C_0(\mathcal{M}(B))$ where $\mathcal{M}(B)$ is the character space of B. Since the spectrum of a hermitian element h in B is real, it follows that any character of the commutative C^*-algebra B is hermitian, that is, the Gelfand transform of x^* for $x \in B$ is given by $\hat{x}^*=\bar{\hat{x}}$ as a function defined on $\mathcal{M}(B)$. For convenient, we state

Proposition 1 (Bourbaki [2, p. 72]). *Suppose that A is a commutative Banach *-algebra, ρ is a canonical morphism of A into $C^*(A)$. Then ρ induces a homeomorphism $\tilde{\rho}$ which maps $\mathcal{M}(C^*(A))$ onto a closed subset H of hermitian characters in $\mathcal{M}(A)$.*

This proposition shows that $\mathcal{M}(A)$ and $\mathcal{M}(C^*(A))$ are different character spaces in general. But in the case $A=L^1(G)$ for a locally compact abelian group G with dual group $\hat{G}$, the character space $\mathcal{M}(L^1(G))\simeq\hat{G}$ and every character of $L^1(G)$ is hermitian, thus (see Bourbaki [2, p. 131])

$$\mathcal{M}(C^*(G))\simeq\mathcal{M}(L^1(G))\simeq\hat{G}.$$

Since the multiplier algebra $M(A)$ of a commutative Banach *-algebra A is a commutative Banach *-algebra with identity, it follows that the enveloping C^*-algebra $C^*(M(A))$ possesses an identity, and so the character space Y of $C^*(M(A))$ is compact and the character space X of $C^*(A)$ is locally compact with respect to the Gelfand topology. If A has an approximate identity, then A is strictly dense in $M(A)$. If A has an identity then $A=M(A)$. Thus

$$C_0(X)\cong\widehat{C^*(A)}\subset\widehat{C^*(M(A))}\cong C_0(Y)=C(Y)$$

where $\hat{B}$ denotes the Gelfand transforms of the algebra B.

We state our main results as follows.

Theorem 2. *Let A be a commutative Banach *-algebra with a bounded approximate identity. Then*

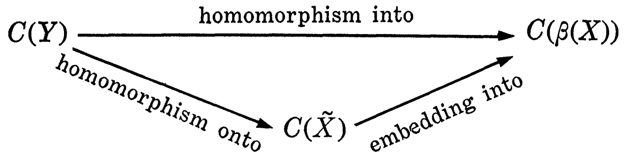
(i) *$C^*(A)$ is an ideal of $C^*(M(A))\cong C(Y)$.*

(ii) *There is a surjective mapping τ,*

$$\tau: \beta(X)\longrightarrow\tilde{X}\subset Y$$

where $\beta(X)$ is the Stone-Čech compactification of X and $\tilde{X}$ is the weak closure of X in Y.

(iii) *The onto mapping τ of* (ii) *induces a homomorphism of $C(Y)$ into $C(\beta(X))$ such that the following diagram commutes*



(iv) *If X is dense in Y, then there exists an isometric embedding ρ of $C^*(M(A))$ into $M(C^*(A))$ so that $C^*(M(A)) \subset M(C^*(A))$.*

Proof. (i) Since A has a bounded approximate identity, A is strictly dense in $M(A)$. It is known that

(1) $$C^*(A) \subset C^*(M(A)) \cong \widehat{C^*(M(A))} \cong C(Y).$$

Now for any $b \in C^*(M(A))$ and $a \in C^*(A)$, there exist sequences $\{b_n\}$ in $M(A)$ and $\{a_m\}$ in A such that $b_n a_m \in A$ for all n, m and

$$ba = \lim_{n\to\infty} b_n a = \lim_{n\to\infty} \lim_{m\to\infty} b_n a_m \in C^*(A).$$

Hence $C^*(A)$ is $C^*(M(A))$ module and $C^*(A)$ is an ideal of $C^*(M(A)) \cong C(Y)$.

(ii) Since we have seen that $C^*(A)$ is an ideal of $C^*(M(A))$, it follows from Lai [7, Theorem 2] that the maximal ideal space X of $C^*(A)$ is homeomorphic to an open subset of the maximal ideal space Y of $C^*(M(A))$. Since Y is compact and X is open in Y, the set $K = Y \setminus X$ is compact in Y. This implies that

$$C^*(A) = \{a \in C^*(M(A))\,;\ \hat{a}|_K = 0\}$$

where $\hat{a}|_K$ is the restriction of the Gelfand transform $\hat{a}$ on K. Since $C^*(A) \cong \widehat{C^*(A)} \cong C_0(X)$, we have

(2) $$M(C^*(A)) \cong M(C_0(X)) \cong C^b(X)$$

where $C^b(X)$ denotes the space of bounded continuous functions on X. Each function in $C^b(X)$ can be extended as a continuous function on the Stone-Čech compactification $\beta(X)$ of X. That is,

$$C^b(X) = C(\beta(X))$$

(see Dunford and Schwartz [3, IV 6.22 and 6.27]). Therefore by [3, 6.27], there is an onto mapping

$$\tau : \beta(X) \longrightarrow \tilde{X}$$

where $\tilde{X}$ is the weak closure of X in Y.

(iii) By (ii), there is an onto mapping $\tau : \beta(X) \to \tilde{X}$, it induces a surjective homomorphism

$$\tilde{\tau} : C(Y) \longrightarrow C(\tilde{X})$$

and an injective embedding

$$j : C(\tilde{X}) \longrightarrow C(\beta(X))$$

such that $\rho = j \circ \tilde{\tau}$ is a homomorphism of $C(Y)$ into $C(\beta(X))$.

(iv) If X is dense in Y, then the onto homomorphism in (iii) becomes an isometric isomorphism. Therefore $\rho = j \circ \tilde{\tau}$ in (iii) is an isometric isomorphism of $C(Y)$ into $C(\beta(X)) = C^b(X)$. It follows from (1) and (2) that

$$\rho: C^*(M(A)) \cong C(Y) \longrightarrow M(C^*(A)) \cong C^b(X)$$

is an isometric injection, that is

$$C^*(M(A)) \subset M(C^*(A)).$$

3. Isometric isomorphism of C^*-algebras. It is known that if a Banach algebra A with a bounded approximate identity is continuously embedded in another Banach algebra B as a dense ideal, then $A=B$ (cf. Lai [7, Proposition 5 and the remark in p. 233]). This reduces that any proper Segal algebra has no bounded approximate identity (cf. also Feichtinger [4, Corollary 2.3]). Hence if a commutative Banach *-algebra A with a bounded approximate identity is an ideal of its enveloping C^*-algebra $C^*(A)=B$, then A itself is a C^*-algebra, it follows that $A=C^*(A)$ as well as $C^*(M(A))=M(A)$ since $M(A)$ is a C^*-algebra with identity provided A is a C^*-algebra. It is remarkable that the maximal ideal space X of $C^*(A)=A$ is dense in the maximal ideal space Y of $C^*(M(A))=M(A)$ with respect to the hull kernel topology if A is semisimple (see Lai [7, § 4]). Hence by Theorem 2 (iv), we have

$$M(A)=C^*(M(A)) \subset M(C^*(A))=M(A) \quad \text{so that} \quad C^*(M(A))=M(C^*(A)).$$

Hence we have the following

Theorem 3. *Let A be a commutative semisimple Banach *-algebra with an approximate identity. If A is an ideal of $C^*(A)$ then*

$$C^*(M(A))=M(C^*(A)).$$

By the above discussion, we see that for a locally compact abelian group G, by Theorem 3, we have

Corollary 4. *Let G be a locally compact abelian group. Then, $L^1(G)$ is an ideal of its enveloping C^*-algebra $C^*(G)$ if and only if G is finite.*

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