100. *Integral Transforms in Hilbert Spaces*

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1. Introduction. We let dm denote a σ finite positive measure and $L_2(dm)$ a usual Hilbert space composed of dm integrable complex valued functions $F(t)$ on a dm measurable set T and with finite norms

$$\|F\|^2_{L_2(dm)}=\int_T |F(t)|^2\, dm(t).$$

For an arbitrary set E and any fixed complex valued function $h(t,p)$ on $T\times E$ satisfying $h(t,p)\in L_2(dm)$ for any fixed $p\in E$, we consider the integral transform of $F\in L_2(dm)$

$$f(p)=\int_T F(t)\overline{h(t,p)}dm(t). \tag{1.1}$$

Then, we first show that the functions $f(p)$ form a Hilbert (possibly finite dimensional) space H which is naturally determined by the integral transform. Furthermore, we establish the fundamental relationship between the two Hilbert spaces $L_2(dm)$ and H.

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2. The image by the integral transform and norm inequality. We define the function $K(p,q)$ on $E\times E$

$$K(p,q)=\int_T h(t,q)\overline{h(t,p)}dm(t). \tag{2.1}$$

Note that $K(p,q)$ is a positive matrix on E in the sense of Moore; i.e.,

$$\sum_{\nu=1}^{m}\sum_{\mu=1}^{m}\alpha_\nu\bar{\alpha}_\mu K(p_\nu,p_\mu)\geqq 0$$

for any finite set $\{p_\nu\}$ of E and for any complex numbers $\{\alpha_\nu\}$. This implies that for $K(p,q)$, there exists a uniquely determined Hilbert space H composed of functions on E admitting $K(p,q)$ as a reproducing kernel [2], p. 344 and [1], p. 143. Then, we obtain

Theorem 1.1. *For the integral transform* (1.1), *we obtain*

$$\|f\|^2_H\leqq\int_T |F(t)|^2\, dm(t). \tag{2.2}$$

Further, (1.1) *gives a mapping from* $L_2(dm)$ *onto* H, *and for any* $f\in H$,

$$\|f\|^2_H=\min\int_T |\tilde{F}(t)|^2\, dm(t) \tag{2.3}$$

where the minimum is taken over all functions $\tilde{F}\in L_2(dm)$ *satisfying*

$$(2.4)\qquad f(p)=\int_T \tilde{F}(t)\overline{h(t,p)}dm(t).$$

Moreover, the family of functions $\{h(t,p)\,|\,p\in E\}$ *is complete in* $L_2(dm)$ *if and only if* (1.1) *gives an isometrical mapping between* $L_2(dm)$ *and* H.

For the proof of this theorem, we can apply the general theory of reproducing kernels using the direct integral theory which is established by L. Schwartz [6], pp. 170–174, but we can obtain the theorem by a quite elementary method, directly.

3. Inverse transform. We consider the inverse transform for (1.1). We assume first that $\{h(t,p)\,|\,p\in E\}$ is complete in $L_2(dm)$ and further we assume that for $f\in H$ and for any $p\in E$,

$$(3.1)\qquad \left(f(q),\int_T h(t,p)\overline{h(t,q)}\,dm(t)\right)_H=\int_T (f(q),\overline{h(t,q)})_H\overline{h(t,p)}dm(t).$$

In particular, for almost all t of T with respect to dm measure

$$(3.2)\qquad \overline{h(t,p)}\in H,$$

$$(3.3)\qquad (f(q),\overline{h(t,q)})_H\in L_2(dm).$$

Then, we have immediately

Theorem 3.1. *We assume that* $\{h(t,p)\,|\,p\in E\}$ *is complete in* $L_2(dm)$ *and* (3.1) *is valid. Then, the inverse for* (1.1) *is given by*

$$(3.4)\qquad F(t)=(f(p),\overline{h(t,p)})_H.$$

The condition (3.1) is, in general, strong. In order to obtain a more general inverse formula, we assume that E is a region Ω in an Euclidean space. We let $\{E_N\}_{N=1}^{\infty}$ be an exhaustion of Ω by compact subsets of Ω. We assume that the norm of H is realized in terms of a σ finite positive measure $d\sigma$ on Ω as follows

$$(3.5)\qquad \|f\|_H^2=\lim_{N\to\infty}\int_{E_N}|f(q)|^2\,d\sigma(q)=\int_\Omega |f(q)|^2\,d\sigma(q).$$

Further, we assume that

(3.6) for all N and for all $p\in E$, $|f(q)h(t,p)h(t,q)|$ is integrable on $(t,q)\in T\times E_N$;

(3.7) for all N, $|f(q)f(q')h(t,q)h(t,q')|$ is integrable on $(q,q',t)\in E_N\times E_N\times T$.

Then, we obtain

Theorem 3.2. *We assume that* $\{h(t,p)\,|\,p\in E\}$ *is complete in* $L_2(dm)$ *and there exists an exhaustion* $\{E_N\}_{N=1}^{\infty}$ *of* Ω *by compact subsets of* Ω *satisfying* (3.5), (3.6) *and* (3.7). *Then,*

$$(3.8)\qquad F(t)=s\text{-}\lim_{N\to\infty}\int_{E_N} f(q)h(t,q)d\sigma(q)$$

in the sense of strong convergence in $L_2(dm)$.

When the family $\{h(t,p)\,|\,p\in E\}$ is not complete in $L_2(dm)$, let, for $f\in H$, $F^*(t)\in L_2(dm)$ be such that $\|F^*\|_{L_2(dm)}=$infimum of $\|\tilde{F}\|_{L_2(dm)}$ for $\tilde{F}$ satisfying (2.4). We assume that for any E_N and for any $F\in L_2(dm)$,

$$(3.9)\qquad |F(t)f(q)h(t,q)| \text{ is integrable on } T\times E_N.$$

Then, we obtain

Theorem 3.3. *We assume that there exists an exhaustion* $\{E_N\}_{N=1}^{\infty}$ *of* Ω *by compact subsets of* Ω *satisfying* (3.5), (3.6), (3.7) *and* (3.9). *Then, we have the inverse formula*

$$F^*(t)=s\text{-}\lim_{N\to\infty}\int_{E_N} f(q)h(t,q)d\sigma(q). \tag{3.10}$$

4. **Expansions of reproducing kernels.** The identity (2.1) enables us to construct many concrete reproducing kernels. However, a crucial point in the application of our theory is to realize the (abstract) Hilbert space H admitting $K(p,q)$ as a reproducing kernel. We can realize the space H by using expansions of $K(p,q)$ which contain the expression (2.1) itself. Cf. Berezanskiĭ [3], Chapter VIII.

Next we show other two general methods which give the expression (2.1).

First, suppose that a concrete Hilbert space H with a reproducing kernel $K(p,q)$ and an isometrical mapping $\tilde{L}: H\to L_2(dm)$ are given. The image of $K(p,q)$ by $\tilde{L}$ is denoted by

$$g_{\tilde{L}}(t,q)=\tilde{L}K(p,q). \tag{4.1}$$

Then, we have the desired identity

$$K(p,q)=\int_T g_{\tilde{L}}(t,q)\,\overline{g_{\tilde{L}}(t,p)}\,dm(t). \tag{4.2}$$

See Shapiro-Shields [5] and Burbea [4]. Then, we obtain

Theorem 4.1. *For the integral transform*

$$f(p)=\int_T F(t)\,\overline{g_{\tilde{L}}(t,p)}\,dm(t) \qquad \text{for } F\in L_2(dm), \tag{4.3}$$

we have the identity

$$\|f\|_H^2=\int_T |f(t)|^2\,dm(t). \tag{4.4}$$

Further, (4.3) *gives the isometrical mapping* $\tilde{L}$, *and the family* $\{g_{\tilde{L}}(t,p)\,|\,p\in E\}$ *is complete in* $L_2(dm)$.

Shapiro-Shields [5] and Burbea [4] obtained miscellaneous concrete identities of type (4.2). Therefore, we can obtain many integral transforms whose inverses are concretely determined. Further, we can obtain completeness theorems for the corresponding $\{g_{\tilde{L}}(t,p)\,|\,p\in E\}$.

Secondary, we can use the Parseval's formula. For example, for $F(x)=1/(x^2+a^2)(a>0)$, we have $F_c=\sqrt{1/2}\,\pi e^{-ax}$ and so we obtain

$$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}=\frac{\pi}{2}\int_0^\infty e^{-ax}e^{-bx}dx=\frac{\pi}{2(a+b)}\,(a,b>0).$$

Here F_c denotes the Fourier cosine transform of F. See [7], p. 180.

5. **One example.** Here we discuss only one integral transform by using our general theory. For $x>0$, we consider the integral transform

(5.1) $$f(x)=\frac{2}{\pi}\int_0^\infty \frac{F(t)\sin xt}{t^2}dt$$

for real valued functions $F(t)$ satisfying

(5.2) $$\int_0^\infty \frac{F(t)^2}{t^2}dt<\infty.$$

Following (2.1), we consider the identity

(5.3) $$\frac{2}{\pi}\int_0^\infty \frac{\sin xt \sin yt}{t^2}dt=\min(x, y) \qquad (x, y>0).$$

See [7], p. 180. Here, $\min(x, y)$ is the reproducing kernel for the Hilbert space composed of all functions $f(x)$ on $(0, \infty)$ such that $f(x)$ is absolutely continuous, $f(0)=0$, $f'(x)\in L_2(0, \infty)$ and with the norm

(5.4) $$\left\{\int_0^\infty f'(x)^2dx\right\}^{1/2}$$

From Theorem 2.1, we obtain the identity

(5.5) $$\int_0^\infty f'(x)^2dx=\frac{2}{\pi}\int_0^\infty \frac{F(t)^2}{t^2}dt.$$

Further, from Theorem 3.2, we obtain the inverse transform for (5.1)

(5.6) $$F(t)=s\text{-}\lim_{N\to\infty} F_N(t)$$

for

(5.7) $$F_N(t)=t\int_{1/N}^N f'(x)\cos xt\,dx$$

in the space satisfying (5.2).

A full paper for this résumé will appear in some journal with miscellaneous concrete examples and applications.

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