98. On the Pathwise Uniqueness of Solutions of One-Dimensional Stochastic Differential Equations of Jump Type

By Takashi KOMATSU
Department of Mathematics, Osaka City University

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1. **Introduction.** Many authors discussed the pathwise uniqueness of solutions for one-dimensional stochastic differential equations of diffusion type

(1) $dX_t=\sigma(X_t)dB_t \qquad B_t$: a Brownian motion.

Among others, Yamada and Watanabe [1] showed that the pathwise uniqueness holds if $|\sigma(x)-\sigma(y)|^2\leqq\rho(|x-y|)$ for all x and y in $\boldsymbol{R}^1$ for an increasing function $\rho(x)$ satisfying $\rho(0)=0$ and

$$\int_{+0}\rho(x)^{-1}dx=\infty.$$

Nakao [2] gave another condition for the pathwise uniqueness.

In the present paper, we shall discuss the pathwise uniqueness of solutions for stochastic differential equations of jump type

(2) $$dX_t=\sigma(X_{t-})dZ_t,$$

where Z_t is a symmetric stable process of exponent α $(0<\alpha<2)$ associated with the generator L defined by

(3) $$Lf(x)=\int[f(x+y)-f(x)-I_{(|y|\leqq 1)}yf'(x)]\,|y|^{-1-\alpha}\,dy.$$

Our results are similar to the result in [1]. For example, in case $1<\alpha<2$, the pathwise uniqueness will be proved under the condition that there exists an increasing function $\rho(x)$ such that $\rho(0)=0$,

$$\int_{+0}\rho(x)^{-1}dx=\infty$$

and $|\sigma(x)-\sigma(y)|^\alpha\leqq\rho(|x-y|)$ for all x and y in $\boldsymbol{R}^1$. An example will be given which shows that the condition is nearly best possible without some additional conditions. But the condition can be relaxed in the case where $1<\alpha<(1+\sqrt{5})/2$ and the function $\sigma(x)$ is uniformly positive.

2. **Main theorems.** Let (Ω, F, P) be a probability space with an increasing family $(F_t)_{t\geqq 0}$ of sub-σ-fields of F. Let Z_t be a one-dimensional symmetric stable process with exponent α whose generator L is given by (3). We suppose $Z_0=0$. The measure

$$p(dt, dz)=\sum_{s\in dt} I_{(\Delta Z_s\in dz\setminus\{0\})}$$

is called the Poisson random measure associated with the process Z_t.

Let $q(dt, dz)=p(dt, dz)-|z|^{-1-\alpha}\,dtdz$ and $\sigma(x)$, a Borel measurable function on $\boldsymbol{R}^1$. We shall consider the equation

$$(4)\qquad X_t=x_0+\int_0^t\int_{|z|\leqq 1}\sigma(X_{s-})zq(ds, dz)+\int_0^t\int_{|z|>1}\sigma(X_{s-})zp(ds, dz),$$

which is simply written as (2).

Theorem 1. *Let* $1<\alpha<2$ *and* $\rho(x)$, *an increasing function on* $[0, \infty)$ *satisfying* $\rho(0)=0$ *and*

$$\int_{+0}\rho(x)^{-1}dx=\infty.$$

If

$$|\sigma(x)-\sigma(y)|^\alpha\leqq\rho(|x-y|)\qquad \textit{for all } x, y \textit{ in } \boldsymbol{R}^1,$$

then there is at most one solution of (4) *for each initial value.*

Proof. Similarly to [1], define a sequence $1=a_0>a_1>\cdots$ by

$$\int_{(a_n,a_{n-1}]}\rho(x)^{-1}dx=n.$$

Choose smooth even functions $\phi_n(x)$ on $\boldsymbol{R}^1$ such that

$$\int_{-\infty}^{+\infty}\phi_n(x)dx=1,$$

$\phi_n(x)=0$ for $|x|\leqq a_n$ or $|x|\geqq a_{n-1}$ and

$$(5)\qquad 0\leqq\phi_n(x)\leqq\frac{1}{n\rho(|x|)}\qquad \text{for } a_n<|x|<a_{n-1}.$$

Set $u(x)=|x|^{\alpha-1}$ and $u_n=u*\phi_n$. Since ϕ_n tends to the δ-function at the origin, the function $u_n(x)$ tends to the function $u(x)$ as $n\to\infty$. We shall show that $Lu_n=c\phi_n$ with a certain constant c independent of n, where L is the operator defined by (3). Let $u^\varepsilon(x)=|x|^{\alpha-1}e^{-\varepsilon|x|}$ $(\varepsilon>0)$ and set $u_n^\varepsilon=u^\varepsilon*\phi_n$. The function u_n^ε belongs to the space $\mathcal{S}(\boldsymbol{R}^1)$ of tempered functions. Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and the inverse Fourier transform respectively. Then $Lf=-c_1\mathcal{F}^{-1}[|\xi|^\alpha\mathcal{F}f]$ for each tempered function f, where $c_1=\pi(\Gamma(\alpha+1)\sin(\alpha\pi/2))^{-1}$. Since $\mathcal{F}u_n^\varepsilon(\xi)=\Gamma(\alpha)\{(\varepsilon+i\xi)^{-\alpha}+(\varepsilon-i\xi)^{-\alpha}\}\mathcal{F}\phi_n(\xi)$ and $|\xi|^\alpha\{(\varepsilon+i\xi)^{-\alpha}+(\varepsilon-i\xi)^{-\alpha}\}$ tends to $2\cos(\alpha\pi/2)$ as $\varepsilon\downarrow 0$ for $\xi\neq 0$, we have

$$\begin{aligned}Lu_n&=\lim_{\varepsilon\downarrow 0}Lu_n^\varepsilon=-\lim_{\varepsilon\downarrow 0}c_1\Gamma(\alpha)\mathcal{F}^{-1}[|\xi|^\alpha\mathcal{F}u_n^\varepsilon(\xi)]\\&=-\lim_{\varepsilon\downarrow 0}c_1\Gamma(\alpha)\mathcal{F}^{-1}[|\xi|^\alpha\{(\varepsilon+i\xi)^{-\alpha}+(\varepsilon-i\xi)^{-\alpha}\}\mathcal{F}\phi_n(\xi)]\\&=-2\pi\alpha^{-1}\cot(\alpha\pi/2)\phi_n=c\phi_n.\end{aligned}$$

Let X_t^1 and X_t^2 be any solutions of (4) with the same initial value. Then we have

$$\begin{aligned}&u_n(X_t^1-X_t^2)-u_n(0)\\&\quad=\int_0^t\int|\sigma(X_s^1)-\sigma(X_s^2)|^\alpha Lu_n(X_s^1-X_s^2)ds\\&\qquad+\int_0^t\int[u_n(X_{s-}^1-X_{s-}^2+(\sigma(X_{s-}^1)-\sigma(X_{s-}^2))z)-u_n(X_{s-}^1-X_{s-}^2)]q(ds, dz).\end{aligned}$$

Set $T_k=\inf\{t\,;|X_t^1-X_t^2|>k\}$. Since

$$|\sigma(x)-\sigma(y)|^\alpha Lu_n(x-y)\leqq c\rho(|x-y|)\phi_n(x-y)\leqq c/n$$

by (5), it follows that

$$E[u_n(X^1_{t\wedge T_k}-X^2_{t\wedge T_k})]\leqq u_n(0)+E\left[\int_0^{t\wedge T_k}\frac{c}{n}ds\right].$$

Therefore we have

$$E[|X^1_{t\wedge T_k}-X^2_{t\wedge T_k}|^{\alpha-1}]=0,$$

because $u_n(x)\to u(x)=|x|^{\alpha-1}$ as $n\to\infty$. Since $P[T_k<t]\to 0$ as $k\to\infty$, we conclude that $X^1_t=X^2_t$ a.e.

Theorem 2. *Let $0<\alpha\leqq 1$ and $\rho(x)$, a function as in Theorem 1. Moreover we assume that $\rho(x)$ is concave. If $|\sigma(x)-\sigma(y)|\leqq\rho(|x-y|)$ is satisfied for all x and y in $\boldsymbol{R}^1$, then the solution of (4) is uniquely determined for each initial value.*

Proof. Let $\psi_n(x)=\sqrt{n/2\pi}\exp(-nx^2/2)$ and set $v(x)=|x|$ and $v_n=v*\psi_n$. As in the proof of Theorem 1, it is proved that there is a positive constant c such that $Lv_n=c|x|^{1-\alpha}*\psi_n$. Let X^1_t and X^2_t be any solutions of (4) with the same initial value. Set $Y_t=X^1_t-X^2_t$ and $T_k=\inf\{t\,;|Y_t|>k\}$. Similarly to the proof of Theorem 1, we have

$$E[v_n(Y_{t\wedge T_k})]\leqq v_n(0)+E\left[\int_0^{t\wedge T_k}\rho(|Y_s|)^\alpha Lv_n(Y_s)\,ds\right].$$

Since $v_n(x)\to|x|$ and $Lv_n\to c|x|^{1-\alpha}$ as $n\to\infty$, it follows that

$$E[|Y_{t\wedge T_k}|]\leqq cE\left[\int_0^{t\wedge T_k}\rho(|Y_s|)^\alpha|Y_s|^{1-\alpha}ds\right].$$

Since $\rho(|y|)^\alpha|y|^{1-\alpha}\leqq\alpha\rho(|y|)+(1-\alpha)|y|$ and $\rho(x)$ is concave, the function $h(t)=E[|Y_{t\wedge T_k}|]$ satisfies the integral inequality

$$h(t)\leqq c\int_0^t[\alpha\rho(h(s))+(1-\alpha)h(s)]ds.$$

This implies that

$$h(t)\leqq c\alpha\int_0^t e^{c(1-\alpha)(t-s)}\rho(h(s))ds.$$

Since

$$\int_{+0}\rho(x)^{-1}dx=\infty,$$

we have $h(t)=0$, and therefore $Y_{t\wedge T_k}=0$ a.e. Hence $Y_t=0$ a.e. because $P[T_k<t]\to 0$ as $k\to\infty$.

3. Complementary results. Let $0<\beta<\alpha\wedge 1$. Then

(6) $$E\left[\int_0^t|Z_s|^{-\beta}ds\right]<\infty.$$

Set

$$T(t)=\int_0^t|Z_s|^{-\beta}ds\quad\text{and}\quad T^{-1}(\tau)=\inf\{t\,;T(t)>\tau\},$$

and define

$$\zeta_\tau=\int_0^{T^{-1}(\tau)}\int_{|z|\leqq|Z_s|^{\beta/\alpha}}|Z_s|^{-\beta/\alpha}zq(ds,dz)+\int_0^{T^{-1}(\tau)}\int_{|z|>|Z_s|^{\beta/\alpha}}|Z_s|^{-\beta/\alpha}zp(ds,dz).$$

The process ζ_τ $(0\leqq\tau<T(+\infty))$ is well defined because of (6). It is easy to show that the process $\{\Omega, F, F_{T^{-1}(\tau)}, P; \zeta_\tau\}$ is an α-stable process with the generator L, and the process $X_\tau=Z_{T^{-1}(\tau)}$ is a solution of the equation: $dX_\tau=|X_{\tau-}|^{\beta/\alpha}d\zeta_\tau$, $X_0=0$. Namely the equation has a non-trivial solution besides the trivial one. Hence we have proved the following.

Proposition. *Consider equation* (4) *for the coefficient* $\sigma(x)=|x|^\gamma$ *and the initial value* $x_0=0$. *Then the pathwise uniqueness holds or not according as* $1\wedge\alpha^{-1}\leqq\gamma$ *or* $0<\gamma<1\wedge\alpha^{-1}$.

This result corresponds with the following one obtained by Girzanov [3] that if $0<\gamma<1/2$, then the stochastic equation $dX_t=|X_t|^\gamma dB_t$ with $X_0=0$ has infinitely many solutions.

Nakao [2] proved that the pathwise uniqueness holds for (1) if the function $\sigma(x)$ is uniformly positive on $\boldsymbol{R}^1$ and if $\sigma(x)$ is of bounded variation on any compact interval.

Theorem 3. *Let* $1<\alpha<2$. *The pathwise uniqueness holds for* (4) *if there are positive constants* λ_1, λ_2, *c and* δ *such that* $\lambda_1\leqq\sigma(x)\leqq\lambda_2$ *for all x in* $\boldsymbol{R}^1$ *and that*

$$|\sigma(x)-\sigma(y)|\leqq c\,|x-y|^{\alpha-1+\delta} \qquad \textit{for all } x, y \textit{ in } \boldsymbol{R}^1.$$

Remark. If $\alpha(\alpha-1)<1$, namely $1<\alpha<(1+\sqrt{5})/2$, this result is not contained in Theorem 1.

Proof of Theorem 3. Define

$$v(x)=\int_0^x \sigma(y)^{-1}dy.$$

Let X_t^1 and X_t^2 be any solutions of (4) with the initial value x_0. Then

$$v(X_t^i)-v(x_0)-Z_t$$
$$=\int_0^t\int\left[\int_0^1\{\sigma(X_{s-}^i)\sigma(X_{s-}^i+\sigma(X_{s-}^i)\theta z)^{-1}-1\}\,d\theta\right]zp(ds, dz).$$

From the assumption the process $V_t=v(X_t^1)-v(X_t^2)$ has the integrable total variation on any compact time interval. The process $M_t=X_t^1-X_t^2$ is a martingale of jump type satisfying

$$\lambda_2^{-1}M_t\leqq V_t\leqq\lambda_1^{-1}M_t \quad \text{as long as} \quad M_t\geqq 0. \tag{7}$$

Using a similar technique to [2], it is proved from property (7) that the martingale M_t is identically zero.

References

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[3] I. V. Girzanov: An example of non-uniqueness of the solution of the stochastic equation of K. Itô. Theory Prob. Appl., 7, 325–331 (1962) (English translation).