97. *Iterating Holomorphic Self-Mappings of the Hilbert Ball*

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Let B denote the open unit ball of a complex Hilbert space H. It has been recently shown [5] that several ideas from the theory of nonexpansive mappings in Banach spaces can be used to yield new results concerning holomorphic self-mappings of B. Continuing in this direction, and motivated by the concept of firmly nonexpansive mappings, we introduce in this note the class of firmly holomorphic self-mappings of B (see the definition below). We show that if a firmly holomorphic $F: B\to B$ has a fixed point, then its iterates $\{F^n x\}$ converge weakly to a fixed point of F for each x in B (Theorem 1). If F is fixed point free, then all its iterates converge strongly to a point on the boundary of B which is independent of x (Theorem 2). We also show how to associate with each holomorphic self-mapping of B family of firmly holomorphic mappings with the same fixed point sets (Theorem 3). We conclude with a discussion of some of the properties of these families (see, for example, Theorem 4).

Recall that a mapping T in a Banach space is said to be firmly nonexpansive [1, 2] if $|Tx-Ty|\le|r(x-y)+(1-r)(Tx-Ty)|$ for all x, y in the domain of T and $r>0$. In this case $|(1-t)(x-y)+t(Tx-Ty)|$ is a (convex) decreasing function for $0\le t\le 1$. Let $\rho: B\times B\to[0,\infty)$ be the hyperbolic metric on B [6]. Since any holomorphic self-mapping of B is nonexpansive with respect to ρ, we shall say that a holomorphic mapping $F: B\to B$ is *firmly holomorphic* if for each x and y in B, the function

$$\rho((1-t)x+tFx,\ (1-t)y+tFy)$$

is decreasing for $0\le t\le 1$.

Let C be a closed convex subset of a Banach space E, and let $T: C\to C$ be a firmly nonexpansive mapping. Assume that both E and its dual E^* are uniformly convex. It is known that if T has a fixed point, then for each x in C, $\{T^n x\}$ converges weakly (but not necessarily strongly) to a fixed point of T. If T is fixed point free, then $\lim_{n\to\infty}|T^n x|=\infty$ for all x in C [3].

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In order to prove an analog of the first result for firmly holomorphic mappings we shall need the following fact. Since ρ-balls are ellipsoids, it is a consequence of the parallelogram law.

Lemma 1. *Let* $\{x_n\}$ *and* $\{z_n\}$ *be two sequences in* B. *Suppose that for some point* $y \in B$, $\lim_{n\to\infty} \rho(x_n, y) = \lim_{n\to\infty} \rho(z_n, y) = \lim_{n\to\infty} \rho((x_n + z_n)/2, y) = d$. *Then* $\lim_{n\to\infty} \rho(x_n, z_n) = 0$.

Theorem 1. *Let* B *denote the open unit ball of a complex Hilbert space. If a firmly holomorphic mapping* $F : B \to B$ *has a fixed point, then for each* x *in* B, *the sequence of iterates* $\{F^n x\}$ *converges weakly to a fixed point of* F.

Proof. Let y be a fixed point of F and let $x_n = F^n x$. Since F is firmly holomorphic, we can apply Lemma 1 to $\{x_n\}$ and $\{Fx_n\}$ to conclude that $\lim_{n\to\infty} \rho(x_n, Fx_n) = 0$. This implies, in turn, that $\{x_n\}$ converges weakly to its asymptotic center.

We do not know if the convergence established in Theorem 1 is actually strong. As mentioned above, this is not true in general in the firmly nonexpansive case. It would also be of interest to determine all the holomorphic self-mappings of B for which Theorem 1 holds.

In order to determine the behavior of a fixed point free $F : B \to B$, we recall [4] that to each holomorphic $T : B \to B$ which is fixed point free we can associate a unique point $e = e(T)$ on the boundary of B with the following property: there is a family of ellipsoids which are invariant under T and whose norm-closures intersect the unit sphere at e. Each such ellipsoid is a set of the form $\{x \in B : \phi_e(x) < a\}$, where $\phi_e(x) = |1-(x, e)|^2/(1-|x|^2)$ and $0 < a < \infty$.

Lemma 2. *Let* $\{x_n\}$ *and* $\{z_n\}$ *be two sequences in* B. *Suppose that* $\lim_{n\to\infty} \phi_e(x_n) = \lim_{n\to\infty} \phi_e(z_n) = \lim_{n\to\infty} \phi_e((x_n + z_n)/2)$. *Then* $\lim_{n\to\infty} |x_n - z_n| = 0$.

Theorem 2. *Let* F *be a firmly holomorphic self-mapping of* B. *If* F *is fixed point free, then for each* x *in* B, *the sequence of iterates* $\{F^n x\}$ *converges strongly to* $e(F)$, *a point on the boundary of* B.

Proof. Let $x_n = F^n x$. Since it can be shown that $\phi_e((1-t)x + tFx)$ is decreasing for $0 \le t \le 1$, we can use Lemma 2 to deduce that $\lim_{n\to\infty} |x_n - Fx_n| = 0$. Since F is fixed point free, it follows that $\lim_{n\to\infty} |x_n| = 1$. Since $\phi_e(x_n) \le \phi_e(x)$ for all n, $\lim_{n\to\infty} (x_n, e) = 1$ and the result follows.

If T is a nonexpansive self-mapping of a closed convex subset C of a Banach space, then for each $0 \le k < 1$ there is a firmly nonexpansive mapping $g_k : C \to C$ that satisfies $g_k(x) = (1-k)x + kTg_k(x)$ for all $x \in C$.

Using the same idea we are now going to associate with each holomorphic mapping $T : B \to B$ a family of firmly holomorphic self-mappings of B with the same fixed point sets. To this end, let $0 \le k$

<1 and fix a point w in B. Define a sequence of holomorphic mappings $f_n: B \to B$ by $f_1(x) = (1-k)x + kTw$, $f_{n+1}(x) = (1-k)x + kT(f_n(x))$, $n \geq 1$. For each fixed $x \in B$, consider the mapping $S: B \to B$ defined by $Sz = (1-k)x + kTz$. Since $|Sz| \leq (1-k)|x| + k < 1$ for all z in B, $\rho(Sz_1, Sz_2) \leq A\rho(z_1, z_2)$ for some $A < 1$. Thus S has a unique fixed point, which we denote by $F(k, T)x$, and $F(k, T)x =$ the strong $\lim_{n\to\infty} S^n w$. In other words, $F(k, T)x = \lim_{n\to\infty} f_n(x)$ for each $x \in B$. Since the sequence $\{f_n(x)\}$ is uniformly bounded, $F(k, T)$ is seen to be a holomorphic self-mapping of B [7, p. 113]. It is clear that T and $F(k, t)$ have the same fixed point sets.

Theorem 3. *Let T be a holomorphic self-mapping of B. For each $0 \leq k < 1$ and $x \in B$ define $Fx = F(k, T)x$ by $Fx = (1-k)x + kTFx$. Then $F: B \to B$ is firmly holomorphic.*

Proof. We already know that F is holomorphic. Now let $0 \leq s < t \leq 1$, $u = (1-s)x + sFx$, $v = (1-t)x + tFx$, $w = (1-s)y + sFy$, and $z = (1-t)y + tFy$. A computation shows that $Fx = F(k, T)x = F(p, T)v$, where $p = k(1-t)/(1-kt)$. We also have $v = (1-q)u + qFx$ and $z = (1-q)w + qFy$, with $q = (t-s)/(1-s)$. Therefore, $v = Gu$ and $z = Gw$, where $G = F(q, F(p, T))$. Since G is holomorphic, we see that $\rho(v, z) = \rho(Gu, Gw) \leq \rho(u, w)$, as required.

Consider once again the firmly nonexpansive mappings g_k mentioned above. It is known [9] that if T has a fixed point, then the strong $\lim_{k\to 1} g_k(x) = Px$ exists for each x in C. P is the unique sunny nonexpansive retraction of C onto the fixed point set of T. In Hilbert space, this retraction coincides with the nearest point projection. If T is fixed point free, then $\lim_{k\to 1} |g_k(x)| = \infty$ for all x in C [8].

Theorem 4. *Let T be a holomorphic self-mapping of B. For each $0 \leq k < 1$ and $x \in B$ define $Fx = F(k, T)x$ by $Fx = (1-k)x + kTFx$. If T is fixed point free, then the strong $\lim_{k\to 1} F(k, T)x = e(T)$, a point on the boundary of B.*

The proof of Theorem 4 resembles that of Theorem 2.

We conjecture that if T has a fixed point, then for each x in B the strong $\lim_{k\to 1} F(k, T)x = Rx$, where R is the nearest point projection (with respect to ρ) from B onto the fixed point set of T. This has been shown to be true in several special cases. Also, R is indeed firmly holomorphic.

It is expected that detailed proofs of the results announced here, as well as other related results, will appear elsewhere.

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