89. On a Certain Property of Profinite Groups

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1. We are led to consider a certain property of profinite groups in investigating a problem posed by Jehne [1] on Kronecker sets of algebraic number fields. Let Q be the rational number field, k a finite algebraic extension of Q and K a finite algebraic extension of k. For the extension K/k, we consider the set $P(K/k)$ of all prime divisors of k having a prime divisor of first relative degree in K/k. We call this set $P(K/k)$ the Kronecker set of K/k. For Kronecker sets, we denote equality of sets up to finite set by $\doteqdot$. Now, the problem of Jehne asks whether there exists a sequence $\{k_n\}_{n=1}^{\infty}$ of finite algebraic extensions of k such that $k_{n+1}\supsetneqq k_n$ and that $P(k_1/k)\doteqdot P(k_n/k)$ for any positive integer n. We call the above sequence $\{k_n\}_{n=1}^{\infty}$ an infinite Kronecker tower of k. In Satz 6 of [2], Klingen claimed that there exists no infinite Kronecker tower of k. As we shall see in the following, the proof of the theorem contains an argument, which is not correct. The following property of Kronecker sets is well-known:

Proposition 1. *Let k be a finite algebraic extension of Q, L a (finite or infinite) Galois extension of k and G the Galois group of L over k. Let H and H' be open subgroups of G. Let K and K' be subfields of L corresponding to the subgroups H and H' of G, respectively. Then the following conditions are equivalent:*

(1) $P(K/k)\doteqdot P(K'/k)$.

(2) $\bigcup_{g\in G} g^{-1}Hg=\bigcup_{g\in G} g^{-1}H'g$.

We owe the following lemma essentially to Klingen [2]:

Lemma 1. *Let L be a (finite or infinite) Galois extension of k and G the Galois group $G(L/k)$ of L over k. For any positive integer n, we denote by k_n a finite algebraic extension of k such that L contains k_n. We suppose that k_{n+1} contains k_n for any positive integer n. Let $K=\bigcup_{n=1}^{\infty} k_n$, let $H=G(L/K)$ and let $H_n=G(L/k_n)$. Then the following conditions are equivalent:*

(1) *$P(k_1/k)\doteqdot P(k_n/k)$ for any positive integer n.*

(2) $\bigcup_{g\in G} g^{-1}H_1g=\bigcup_{g\in G} g^{-1}Hg$.

The following lemma follows immediately from the fact that G is a profinite group:

Lemma 2. *Let L be a (finite or infinite) Galois extension of k and K an intermediate field L over k. Let $G=G(L/k)$ and $H=G(L/K)$.*

If $\bigcup_{g\in G} g^{-1}Hg$ is an open subset of G, then there exists a finite Galois extension F of k such that L contains F and that $\bigcup_{g\in G} g^{-1}G(L/FK)g = G(L/F)$.

Lemmas 1 and 2 yield the following

Theorem 1 (cf. [2]). *Let k be a finite algebraic extension of Q, L a (finite or infinite) Galois extension of k and G the Galois group of L over k. Then the following conditions are equivalent:*

(1) *The Galois group G has a property that $\bigcup_{g\in G} g^{-1}Hg$ is not open in G for any non-open closed subgroup H of G.*

(2) *There exists no infinite Kronecker tower $\{k_n\}_{n=1}^{\infty}$ of k, such that L contains $\bigcup_{n=1}^{\infty} k_n$.*

For convenience's sake, we shall use the following definition:

Definition. A profinite group G is called regular, if for any non-open closed subgroup H of G, $\bigcup_{g\in G} g^{-1}Hg$ is not open in G.

2. Let k be a finite algebraic extension of Q, $\bar{k}$ the algebraic closure of k and G_k the Galois group of $\bar{k}$ over k. In the proof of Satz 6 of [2], it was claimed that G_k is regular from the ground that for a non-open closed subgroup H of G_k, the group index $(G_k : H)$ should be countable. This is, however, not the case, as the following shows:

Proposition 2. *Let G be a compact group and H a non-open closed subgroup of G. Then the group index $(G:H)$ is not countable.*

Proof. Let μ be a Haar measure of G such that $\mu(G)=1$. Suppose that $(G:H)$ is countable. Let $G=\bigcup_{i=1}^{\infty} Hg_i$ be the disjoint union of the right cosets of H. We have $\mu(G)=\sum_{i=1}^{\infty}\mu(Hg_i)$ and $\mu(Hg_i) = \mu(H)$. If $\mu(H)>0$, then we have $\mu(G)=\infty$. If $\mu(H)=0$, then we have $\mu(G)=0$. This is a contradiction.

3. Now, we shall show that some profinite groups are regular.

Lemma 3. *Let X be a topological space, n a positive integer and X_i a non-empty closed subset of X for $i=1, 2, \cdots, n$. If $\bigcup_{i=1}^{n} X_i$ is an open subset of X, then there exists an integer i_1 $(1\le i_1\le n)$ such that X_{i_1} contains a non-empty open subset of X.*

Proof. Let $W=\bigcup_{i=1}^{n} X_i$ and $m=\min\{l \mid W=\bigcup_{\nu=1}^{l} X_{j_\nu}\}$. Let $W = \bigcup_{\nu=1}^{m} X_{i_\nu}$. Then X_{i_1} contains the non-empty open subset $W-\bigcup_{\nu=2}^{m} X_{i_\nu}$ of X.

Theorem 2. *Let G be a profinite group and N a closed abelian normal subgroup of G. If the factor group G/N is regular, then G is regular.*

Proof. Let H be a closed subgroup of G such that $\bigcup_{g\in G} g^{-1}Hg$ is an open subset of G. Since G/N is regular and since the set $\bigcup_{g\in G}(g^{-1}HgN)/N$ is an open subset of G/N, the factor group HN/N is an open subgroup of G/N. Hence HN is an open subgroup of G. Let $G=\bigcup_{i=1}^{n} HNg_i$ be the disjoint union of the right cosets of HN. Since N is a normal abelian subgroup of G, we have

$$N \cap \left(\bigcup_{g \in G} g^{-1}Hg\right) = \bigcup_{g \in G} g^{-1}(N \cap H)g = \bigcup_{i=1}^{n} g_i^{-1}(N \cap H)g_i.$$

Hence, from Lemma 3, we see that $H \cap N$ is an open subgroup of N. Since the group index $(G:H)$ is equal to $(G:HN)(N:N \cap H)$, H is an open subgroup of G. Hence we see that G is regular.

Theorem 3. *Let G be a profinite group such that G contains an open normal solvable subgroup of G. Then G is regular.*

Proof. There exists a sequence $\{N_i\}_{i=0}^{r}$ of normal closed subgroup of G, which satisfies the following conditions:

(1) $N_0 \supset N_1 \supset \cdots \supset N_r$.

(2) N_0 is an open normal subgroup of G.

(3) N_i/N_{i+1} is an abelian group for $i=0, 1, \cdots, r-1$.

(4) $N_r=\{1\}$.

We use induction on r. If $r=0$, then G is a finite group, which shows that G is regular. If $r>0$, by applying the induction, we see that G/N_{r-1} is regular, Then, from Theorem 2, we see that G is regular.

4. Now we shall show the following

Theorem 4. *A pro-nilpotent group is regular.*

For the proof we need the following

Lemma 4. *Let G be a nilpotent group, N a normal subgroup of G and H a subgroup of N. If $N=\bigcup_{g \in G} g^{-1}Hg$, then we have $H=N$.*

Proof. There exists a sequence $\{N_i\}_{i=0}^{r}$ of normal subgroups of G satisfying the following conditions:

(1) $N_0 \supset N_1 \supset \cdots \supset N_r=\{1\}$.

(2) $N_0=N$.

(3) N_i/N_{i+1} is contained in the center of G/N_{i+1} for $i=0, 1, \cdots, r-1$. We use induction on r. If $r=0$, it is clear that $H=N$. If $r>0$, by applying the induction assumption, we have $HN_{r-1}=N$. Since we have

$$N_{r-1} = \left(\bigcup_{g \in G} g^{-1}Hg\right) \cap N_{r-1} = \bigcup_{g \in G} g^{-1}(H \cap N_{r-1})g = H \cap N_{r-1},$$

we have $H=N$.

Proof of Theorem 4. Let G be a pro-nilpotent group and H a closed subgroup of G such that $\bigcup_{g \in G} g^{-1}Hg$ is an open subset of G. Then there exists an open normal subgroup N of G such that $\bigcup_{g \in G} g^{-1}Hg \supset N$. Let $H_1=N \cap H$. We have $N=\bigcup_{g \in G} g^{-1}H_1g$. Let N_α be any open normal subgroup of G such that $N \supset N_\alpha$. Then we have $N/N_\alpha=\bigcup_{g \in G} (g^{-1}H_1gN_\alpha)/N_\alpha$, which shows $N=H_1N_\alpha$ from Lemma 4. Hence we have $N=H_1$.

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References

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[2] N. Klingen: Zahlkörper mit gleicher Primzerlegung. J. reine angew. Math., **299/300**, 342–384 (1978).