84. On Involutive Systems of Second Order of Codimension 2*)

By Keizo YAMAGUCHI
Department of Mathematics, Hokkaido University

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In [1] and [2], E. Cartan obtained, among others, the following result (p. 2 [2]), for which he gave the proof in the case of 2 or 3 independent variables:

Tout système en involution de deux équations aux dérivées partielles du second ordre s'intègre par des équations différentielles ordinaires.

The purpose of this note is to give a precise statement of the above theorem and to describe methods of integration in the case of n independent variables ($n \geqq 4$). Details will be published elsewhere. In this note, we always assume the differentiability of class C^∞.

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§1. Classification of symbols and the statement of Theorem. Let (M, N, p) be a fibred manifold of fibre dimension 1, where $\dim N = n$ and $\dim M = n+1$. Let $J^k(M, N, p)$ be the bundle of k-jets of local sections of (M, N, p) and C^k be the canonical differential system on $J^k(M, N, p)$.

Let R be an involutive system of second order which is locally defined by two equations, i.e., R is an involutive submanifold of $J^2(M, N, p)$ of codimension 2. Let x_0 be any point of R. Our problem is to find every local solution f of R passing through x_0. More precisely, f is a section of (M, N, p) defined on a neighborhood U' of $z_0 = p^2_{-1}(x_0)$ such that $j^2_{z_0}(f) = x_0$ and $j^2(f)(U') \subset R$.

First we have

Proposition 1 (cf. p. 11 [2]). *Let V be a vector space* (*over* $\boldsymbol{R}$ *or* $\boldsymbol{C}$) *of dimension n. Let A be a subspace of $S^2(V^*)$ of codimension* 2. *Then A is involutive if and only if there exists a base $\{e_1, \cdots, e_n\}$ of V such that the annihilator $A^\perp$ of A in $S^2(V)$ is generated by $e_1 \circledcirc e_2$ and $e_1 \circledcirc e_3$, or $e_1 \circledcirc e_1$ and $e_1 \circledcirc e_2$.*

Let $\mathfrak{C}^2(V, W)$ be the contact algebra of second order of degree n (Definition 3.5 [3]). We now define involutive subalgebras $\mathfrak{s}^1$ and $\mathfrak{s}^2$ of $\mathfrak{C}^2(V, W)$ of codimension 2 by putting

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$$\mathfrak{s}^i = \mathfrak{s}^i_{-3} \oplus \mathfrak{s}^i_{-2} \oplus \mathfrak{s}^i_{-1} \qquad (i=1,2),$$
$$\mathfrak{s}^i_{-3} = W, \quad \mathfrak{s}^i_{-2} = W \otimes V^*, \quad \mathfrak{s}^i_{-1} = V \oplus W \otimes \mathfrak{f}^i \quad (i=1,2),$$
where $(f^1)^\perp = \langle e_1 \circledcirc e_2,\ e_1 \circledcirc e_3 \rangle$, $(f^2)^\perp = \langle e_1 \circledcirc e_1,\ e_1 \circledcirc e_2 \rangle \subset S^2(V)$.

From Proposition 1, it follows that $(R; D^1, D^2)$ is a *PD* manifold of second order of degree n satisfying $D^1 = \partial D^2$, where $D^1 = \partial C^2|_R$, $D^2 = C^2|_R$ and ∂D^2 is the derived system of D^2 (cf. § 4 [3]). Furthermore the symbol algebra $\mathfrak{s}(x)$ of $(R; D^1, D^2)$ at $x \in R$ is isomorphic with $\mathfrak{s}^1$ or $\mathfrak{s}^2$. Our regularity assumption is that R is regularly involutive, i.e., symbol algebras $\mathfrak{s}(x)$ are mutually isomorphic for $x \in R$ (§ 5 [3]).

Now the main theorem can be stated as follows.

Theorem. *Let R be an involutive system of $J^2(M, N, p)$ of codimension 2 ($n \geqq 4$). Assume that R is regularly involutive on a neighborhood of $x_0 \in R$. Then every local solution of R passing through x_0 can be obtained by solving ordinary differential equations.*

From now on, let R be a regularly involutive submanifold of $J^2(M, N, p)$ of codimension 2. Then (R, D^2) is a regular differential system of type $\mathfrak{s}^1$ or $\mathfrak{s}^2$. In both cases, since $E = \langle e_1 \rangle$ is $G(\mathfrak{s}^i)$-invariant, we can consider the first order covariant system $N = N(E)$ of D^2 corresponding to E (see § 7 [3]).

§ 2. Structure equations. Let x_0 be any point of R. Then there exists a base $\{\pi, \pi_i, \omega_i, \pi_{ij}\ (1 \leqq i \leqq j \leqq n)\}$ of 1-forms on a neighborhood U of x_0 such that $D^2 = \{\pi = \pi_1 = \cdots = \pi_n = 0\}$ and that the following equalities hold (cf. Remark 6.7 (2) [3]);

$$\begin{cases} d\pi \equiv \sum_{i=1}^n \omega_i \wedge \pi_i & (\text{mod } \pi), \\ d\pi_i \equiv \sum_{j=1}^n \omega_j \wedge \pi_{ij} & (\text{mod } \pi, \pi_1, \cdots, \pi_n), \end{cases}$$

where $\pi_{ij} = \pi_{ji}\ (1 \leqq i, j \leqq n)$ and $\pi_{12} = \pi_{13} = 0$ (case of type $\mathfrak{s}^1$) or $\pi_{11} = \pi_{12} = 0$ (case of type $\mathfrak{s}^2$).

A base $\{\pi, \pi_i, \omega_i, \pi_{ij}\ (1 \leqq i \leqq j \leqq n)\}$ of 1-forms on U satisfying $D^2 = \{\pi = \pi_1 = \cdots = \pi_n = 0\}$ is called admissible if it satisfies the above equations. In both cases N is defined on U by π and π_1. The structure equation of N is given as follows.

Lemma 1. *If R is of type $\mathfrak{s}^1$, there exists an admissible base of 1-forms on U such that the following equalities hold:*

$$\begin{cases} d\pi \equiv \sum_{i=2}^n \omega_i \wedge \pi_i & (\text{mod } \pi, \pi_1), \\ d\pi_1 \equiv \omega_1 \wedge \pi_{11} + \sum_{\alpha=4}^n (\omega_\alpha + \sum_{\beta=4}^n A_{\alpha\beta} \pi_\beta) \wedge \pi_{1\alpha} & (\text{mod } \pi_1), \end{cases}$$

where $A_{\alpha\beta}$ are functions on U such that $A_{\alpha\beta} + A_{\beta\alpha} = 0$.

Lemma 2. *If R is of type $\mathfrak{s}^2$, there exists an admissible base of 1-forms on U such that the following equalities hold:*

$$\begin{cases} d\pi \equiv \sum_{i=2}^n \omega_i \wedge \pi_i & (\text{mod } \pi, \pi_1), \\ d\pi_1 \equiv \sum_{\alpha=3}^n \omega_\alpha \wedge \pi_{1\alpha} + B\omega_2 \wedge \pi_3 + C\pi_2 \wedge \pi_{22} & (\text{mod } \pi, \pi_1), \end{cases}$$

where B and C are functions on U. Furthermore $B \cdot C = 0$ and $C = 0$ if $n \geqq 4$.

By Lemma 1, we see that if R is of type $\mathfrak{F}^1$, $N^*=\{\pi_1=0\}$ is a covariant system of N. Furthermore, from Lemmas 1 and 2, it follows that $\nu_x(\mathrm{Ch}\,(N)(x)\cap \mathrm{Ch}\,(D^1)(x))=\{a\in S^2(V^*)\,|\,v\lrcorner a=0 \text{ for } v\in E\}$ in both cases, where ν_x is an isomorphism of the symbol algebras $\mathfrak{F}(x)$ onto $\mathfrak{F}^i$.

§3. Reduction of (R, D^2). For each $u\in J^1(M,N,p)$, let I_u be the set of all hyperplanes in $C^1(u)$. We now consider the (involutive) Grassmann bundle $I(\mathfrak{C},1)$ of codimension 1 over $\mathfrak{C}=(J^1(M,N,p),C^1)$:

$$I(\mathfrak{C},1)=\bigcup_{u\in J^1(M,N,p)} I_u.$$

Let φ be a map of R into $I(\mathfrak{C},1)$ defined by

$$\varphi(x)=\rho_*(N(x)),$$

where ρ is the projection of R onto $J^1(M,N,p)$. Then we see that φ is a map of constant rank and $\mathrm{Ker}\,\varphi_*=\mathrm{Ch}\,(N)\cap\mathrm{Ch}\,(D^1)$.

In the following we restrict our considerations in a neighborhood of $x_0\in R$ so that we may assume that $W=\mathrm{Im}\,\varphi$ is a submanifold of $I(\mathfrak{C},1)$. Then φ is a submersion of R onto W satisfying $\rho=q\cdot\varphi$, where q is the projection of W onto $J^1(M,N,p)$. There are two differential systems C and $\bar{N}$ on W such that $N=\varphi_*^{-1}(\bar{N})$ and $D^1=\varphi_*^{-1}(C)$. Put $\psi(x_0)=\varphi_*(D^2(x_0))\subset\bar{N}(w_0)$, $w_0=\varphi(x_0)$. We say that S is a solution of W if S is an n-dimensional integral manifold of $(W,\bar{N})$ such that $T_w(S)\cap\mathrm{Ker}\,p_*=\{0\}$ at each $w\in S$. Now we have

Proposition 2. *For a local solution f of R on a neighborhood U' of $z_0=p^2_{-1}(x_0)$ such that $j^2_{z_0}(f)=x_0$, $s=\varphi\circ j^2(f)(U')$ is a solution of W satisfying $T_{w_0}(S)\subset\psi(x_0)$.*

Conversely, for a solution S of W satisfying $T_{w_0}(S)\subset\psi(x_0)$, there exists a local solution f of R such that $\varphi\circ j^2(f)(U')$ coincides with S around w_0.

Thus our problem is reduced to that of finding every local solution S of W passing through w_0 such that $T_{w_0}(S)\subset\psi(x_0)$. One can also see that the local equivalence problem of (R,D^2) is reduced to that of $(W;C,\bar{N})$.

§4. Method of integration. 4.1. *Case of type $\mathfrak{F}^1$.* Since $\mathrm{Ker}\,\varphi_*=\mathrm{Ch}\,(N)$ and N^* is a covariant system of N, there exists a covariant system $\bar{N}^*$ of $\bar{N}$ such that $N^*=\varphi_*^{-1}(\bar{N}^*)$. Now the integration is carried out by the following two steps (1) and (2).

(1) *Find a maximal integral manifold Σ of $(W,\bar{N}^*)$ such that $w_0\in\Sigma$ and $T_{w_0}(\Sigma)\cap\mathrm{Ch}\,(C)(w_0)=\{0\}$.*

Then $\dim\Sigma=2n$ and $q:\Sigma\to J^1(M,N,p)$ is an immersion around w_0. Hence $\Sigma'=q(\Sigma)$ is a submanifold of $J^1(M,N,p)$ around $u_0=p^2_1(x_0)$.

(2) *Find a local solution f of the first order partial differential equation Σ such that $j^2_{z_0}(f)=x_0$.*

Then f is a local solution of R. Conversely, every local solution f of R passing through x_0 can be obtained in this way.

4.2. *Case of type* $\mathfrak{S}^2$. Ch $(\bar{N})$ is a differential system of rank 1 such that $\mathrm{Ch}\,(\bar{N}) \cap \mathrm{Ch}\,(C) = \{0\}$. And we see that every local solution S of W satisfying $T_{w_0}(S) \subset \psi(x_0)$ is foliated by integral curves of Ch $(\bar{N})$. Let H be a hypersurface in the base space N such that $z_0 = p^2_{-1}(x_0) \in H$ and $T_{z_0}(H) \cap (p^2_{-1})_* (\mathrm{Ch}\,(N)(x_0)) = \{0\}$. Then $\hat{H} = (\mathfrak{p} \circ p^1_{-1})^{-1}(H)$ is a hypersurface of W passing through w_0 which is transversal to Ch $(\bar{N})$ around w_0. Hence, for every solution S of W satifying $T_{w_0}(S) \subset \psi(x_0)$, $S' = S \cap \hat{H}$ is an $(n-1)$-dimensional integral manifold of $(W, \bar{N})$. We call S' an *initial manifold* of S. Thus our problem in this case is to find every initial manifold S' in $\hat{H}$ passing through w_0. Now the integration of S' is carried out by the following steps.

(1) *Find a maximal integral manifold* Σ *of* $(\hat{H}, \hat{C})$ *such that* $w_0 \in \Sigma$ *and* $\bar{N}(w_0) \cap T_{w_0}(\Sigma) = \psi(x_0) \cap T_{w_0}(\hat{H})$, *where* $\hat{C} = C|_H$.

Then we see that $\dim \Sigma = 2n-2$ and that $\hat{N} = \bar{N}|_\Sigma$ is a differential system of codimension 1 on Σ such that Ch $(\hat{N})$ is a subbundle of $\hat{N}$ of codimension $2(n-2)$ around w_0.

(2) *Find a maximal integral manifold* S' *of* $(\Sigma, \hat{N})$ *such that* $w_0 \in S'$ *and* $T_{w_0}(S') \subset \mathrm{Ch}\,(C)(w_0) = \{0\}$.

Then S' is an initial manifold in $\hat{H}$. Conversely every initial manifold S' can be obtained in this way.

References

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