# 82. On the Isomonodromic Deformation for Linear Ordinary Differential Equations of the Second Order. II

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§ 1. **Introduction.** In a previous note [1], we derived a series of six Hamiltonian systems

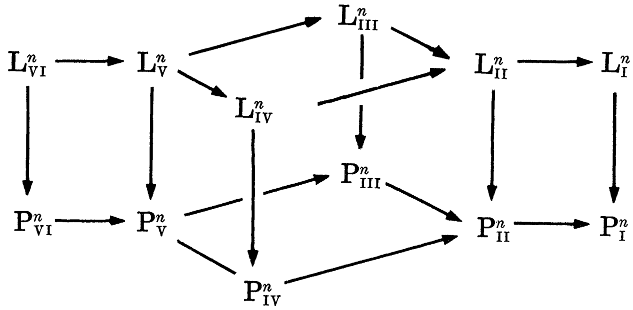
$\mathrm{P}_J^n$ $\qquad d\lambda/dt=\partial H_J^n/\partial\mu, \qquad d\mu/dt=-\partial H_J^n/\partial\lambda$

from a series of six linear differential equations

$\mathrm{L}_J^n$ $\qquad d^2y/dx^2+p_J^n dy/dx+q_J^n y=0,$

where $J=\mathrm{VI}, \mathrm{V}, \cdots, \mathrm{I}$ and $n=1,2,\cdots$, and for $n=1,2,3$, we obtained the commutative diagrams

$\Gamma^n$




Here, by the horizontal arrows above and below we mean processes of confluence of singularities and of degeneration of systems respectively and by the vertical arrows a process of deriving deformation equations.

In this note, we announce the existence of a transformation from $\Gamma^1$ to $\Gamma^2$.

§ 2. **Preliminaries.** If a linear differential equation

(2.1) $$y''+A_1y'+A_2y=0$$

is transformed into an equation

(2.2) $$z''+B_1z'+B_2z=0$$

by a linear transformation

(2.3) $$z=S_1y'+S_2y,$$

then we have the following relations

(2.4) $$S_1''+(B_1-2A_1)S_1'+2S_2'+A_1(A_1-B_1)S_1 +(B_2-A_2-A_1')S_1+(B_1-A_1)S_2=0,$$

(2.5) $S_2''-2A_2S_1'+B_1S_2'+A_2(A_1-B_1)S_1-A_2'S_1+(B_2-A_2)S_2=0.$

Suppose that $A_1$, $A_2$, $B_1$ and $B_2$ are rational functions in $x$. Then the existence of the transformation (2.3) with rational $S_1$, $S_2$ is equivalent to the condition that the monodromy data of (2.1) and (2.2) coincide with each other.

**§3. Equations $L_I^1$, $L_I^2$ and systems $P_I^1$, $P_I^2$.** We denote the equations $L_I^1$ and $L_I^2$ by

(3.1) $$y''-\frac{1}{x-\lambda}y'+\left(-4x^3-2tx-2H-\frac{\mu}{x-\lambda}\right)y=0,$$

(3.2) $$z''-\frac{1}{x-l}z'+\left(-4x^3-2tx-2h-\frac{m}{x-l}\right)z=0$$

and the systems $P_I^1$ and $P_I^2$ by

(3.3) $$d\lambda/dt=\mu, \qquad d\mu/dt=6\lambda^2+t,$$

(3.4) $$\frac{dl}{dt}=\frac{m}{4}+\frac{6l^2+t}{m^2}, \qquad \frac{dm}{dt}=6l^2+t+\frac{12l}{m}.$$

We suppose that to given values of $l, m, h$ there correspond values of $\lambda, \mu, H$ such that the monodromy data of (3.1) coincide with those of (3.2), namely, (3.1) is taken into (3.2) by a transformation of the form (2.3) with rational $S_1$ and $S_2$. Then studies of local behavior of solutions of (3.1) and (3.2) around singularities lead us to

$$S_1=\frac{1}{x-\lambda}, \qquad S_2=s-\frac{\mu}{x-\lambda}.$$

Putting

$$A_1=-1/(x-\lambda), \qquad A_2=-4x^3-2tx-2H+\mu/(x-\lambda),$$
$$B_1=-2/(x-l), \qquad B_2=-4x^3-2tx-2h+m/(x-l),$$

we obtain from (2.4)

(3.5) $$2(\lambda-l)s-2\mu-m=0,$$

(3.6) $$(\lambda-l)s-2(H-h)(\lambda-l)+2\mu+m=0,$$

and from (2.5)

(3.7) $$m(\lambda-l)s+4H+m\mu+4(2l^3+tl)=0,$$

(3.8) $$2H-\mu^2-2(2\lambda^3+t\lambda)=0,$$

(3.9) $$(\lambda-l)\mu s+2(\lambda-l)\mu(H-h)+4H+m\mu+4(2\lambda^3+t\lambda)=0,$$

(3.10) $$(H-h)s-2\lambda-4l=0.$$

The equality (3.8) shows that $x=\lambda$ is an apparent singularity of (3.1). We get from (3.5)–(3.7)

(3.11) $$s=-(\mu+m/2)/(\lambda-l),$$

(3.12) $$H-h=-(\mu+m/2)/2(\lambda-l),$$

(3.13) $$H=m^3/8-(2l^3+tl),$$

and we see that (3.9) is derived from (3.6) and (3.8). Inserting (3.11)–(3.13) into (3.8) and (3.10), we get two equations with respect to $\lambda$ and $\mu$, from which we obtain

(3.14) $$\lambda=-2l+\frac{(6l^2+t)^2}{m}, \qquad \mu=-\frac{m}{2}+\frac{6l(6l^2+t)}{m}-\frac{2(6l^2+t)^3}{m^3}.$$

It is easy to see that the transformation (3.14) changes (3.3) into (3.4) and that (3.14) is a canonical transformation.

**§4. Statement of main theorems and a conjecture.** The arguments and calculations done for $L_I^1$, $L_I^2$, $P_I^1$ and $P_I^2$ can be applied to $L_J^1$, $L_J^2$, $P_J^1$ and $P_J^2$ for $J=\mathrm{II}, \cdots, \mathrm{VI}$.

We write $L_J^1$ and $L_J^2$ as

$$y''+p_J^1 y'+q_J^1 y=0, \tag{4.1}$$

$$z''+p_J^2 z'+q_J^2 z=0 \tag{4.2}$$

and write $P_J^1$ and $P_J^2$ as

$$d\lambda/dt=\partial H_J^1/\partial\mu, \qquad d\mu/dt=-\partial H_J^1/\partial\lambda, \tag{4.3}$$

$$dl/dt=\partial H_J^2/\partial m, \qquad dm/dt=-\partial H_J^2/\partial l. \tag{4.4}$$

**Theorem 1.** *For each $J$ there exists a linear transformation $\sigma_J$ of the form* (2.3) *which takes* (4.1) *into* (4.2) *if and only if $\lambda$ and $\mu$ are related to $l, m, t$ by*

$$\tau_J:\ \lambda=\phi_J(l, m, t), \quad \mu=\psi_J(l, m, t),$$

*where $\phi_J$ and $\psi_J$ are rational functions in $l, m, t$.*

**Theorem 2.** *The transformation $\tau_J$ changes* (4.3) *into* (4.4) *and is a canonical transformation.*

Theorems 1 and 2 say that there exists a transformation

$$\rho:\Gamma^1\longrightarrow\Gamma^2$$

which consists of $\sigma_J$ and $\tau_J$ ($J=\mathrm{VI}, \mathrm{V}, \cdots, \mathrm{I}$).

**Theorem 3.** *The general solution of $P_J^2$ ($J=\mathrm{VI}, \cdots, \mathrm{I}$) is finitely many valued around its fixed singularities and its movable branch points are all algebraic ones.*

**Conjecture.** *For each $n=2, 3, \cdots$, there exists a transformation*

$$\rho^n:\Gamma^1\longrightarrow\Gamma^n$$

*which consists of linear transformations $\sigma_J^n$ and canonical transformations $\tau_J^n$, where $\sigma_J^n$ sends $L_J^1$ into $L_J^n$ and $\tau_J$ sends $P_J^1$ into $P_J^n$. For $n=3, 4, \cdots, \tau_J^n$ are algebraic transformations.*

It is not difficult to verify the existence of $\sigma_I^3$ and $\tau_I^3$.

**§5. Linear transformations $\sigma_J$.** Explicit expressions of $\tau_J$ becomes complicated as $J$ increases, but explicit expressions of $\sigma_J$ remain simple. So we are content to give a list of $\sigma_J$.

$$\sigma_{\mathrm{II}}:S_1=1/(x-\lambda), \qquad S_2=s-\mu/(x-\lambda),$$

$$\sigma_{\mathrm{III}}:S_1=x^2/(x-\lambda), \qquad S_2=s-\lambda^2\mu/(x-\lambda),$$

$$\sigma_{\mathrm{IV}}:S_1=x/(x-\lambda), \qquad S_2=s-\lambda\mu/(x-\lambda),$$

$$\sigma_{\mathrm{V}}:S_1=x(x-1)^2/(x-\lambda),$$
$$S_2=\chi(x-\lambda)+s-\lambda(\lambda-1)^2\mu/(x-\lambda),$$

$$\sigma_{\mathrm{VI}}:S_1=x(x-1)(x-t)/(x-\lambda),$$
$$S_2=\chi(x-\lambda)+s-\lambda(\lambda-1)(\lambda-t)\mu/(x-\lambda).$$

By the way we want to make a comment on the order of the systems $P_I^n, P_{II}^n, \cdots, P_{VI}^n$. If $J$ increases according to the complexity of the systems, then the numbers III and IV should be interchanged.

§ 6. **Calculations by computer.** To prove that $\tau_{\mathrm{VI}}$ is a canonical transformation, it is sufficient to show that

(6.1) $$\partial_x\phi_{\mathrm{VI}}\cdot\partial_y\psi_{\mathrm{VI}}-\partial_x\psi_{\mathrm{VI}}\cdot\partial_y\phi_{\mathrm{VI}}=1.$$

A straightforward examination of (6.1) needs enormous calculations which exceed a scope of handworks. The author thanks Prof. Yasumasa Kanada, Computer Center, University of Tokyo, who checked (6.1) by utilizing a computer algebra system: HLISP-REDUCE-2. The computer was used to perform polynomial multiplications and factorizations. The author thanks also Prof. Eiichi Goto, Department of Information Science, University of Tokyo, for his valuable advices.

## Reference

[1] Kimura, T.: On the isomonodromic deformation for linear ordinary differential equations of the second order. Proc. Japan Acad., 57A, 285–290 (1981).