76. A Uniqueness Result for the Semigroup Associated with the Hamilton-Jacobi-Bellman Operator

By P. L. LIONS*) and Makiko NISIO**)

(Communicated by Kôsaku YOSIDA, M. J. A., Sept. 13, 1982)

1. Introduction. We consider controlled diffusion processes of the form;

$$(1)\qquad \begin{cases} dX(t)=\sigma(X(t),\, v(t))dB_t+b(X(t),\, v(t))dt \\ X(0)=x\in R^N \end{cases}$$

where B_t is an n-dimensional Brownian motion in some probability space (Ω, F, F_t, P), equipped with a filtration satisfying the usual conditions, $\sigma(x, v)$(resp. $b(x, v)$) is an $N\times n$ matrix-valued (resp. N-vector-valued) function on $R^N\times V$ and V is a separable metric space. Precise assumptions on σ, b will be made later on.

The control v is any progressively measurable process with respect to F_t taking its value in a compact subset of V. We introduce a cost function of the form:

$$(2)\qquad J(x, t, \phi, v(\cdot))=E\int_0^t f(X(s),\, v(s))\exp\left(-\int_0^s c(X(\lambda),\, v(\lambda))d\lambda\right)ds + \phi(X(t))\exp\left(-\int_0^t c(X(s),\, v(s))ds\right)$$

where $f(x, v)$, $c(x, v)$ and $\phi(x)$ are real valued functions.

We will always assume: $\exists C>0$ such that

$$(3)\qquad \begin{cases} \|D_x^\alpha\psi\|_{L^\infty(R^N)}\le C,\ \forall v\in V,\ \forall|\alpha|\le 2,\ \forall\psi=\sigma,\ b,\ f,\ c. \\ \psi(x,v) \text{ is continuous in } v,\ \forall x\in R^N,\ \forall\psi=\sigma,\ b,\ f,\ c. \end{cases}$$

(4) $\phi\in X=BUC(R^N)=\{v\in C_b(R^N),\ v \text{ is uniformly continuous on } R^N\}$.

Finally we set

$$(5)\qquad J(x, t, \phi)=\inf_{v(\cdot)} J(x, t, \phi, v(\cdot))$$

where the infinimum is taken over all controls $v(\cdot)$ defined above. We also denote by $(S_0(t)\phi)(x)=J(x, t, \phi)$.

Then, we know (see A. Bensoussan-J. L. Lions [2], N. V. Krylov [6], M. Nisio [11]) that the mathematical formulation of the *dynamic programming principle* is the following:

i) $S_0(t)$ is a semigroup on X.

In addition, one knows that S_0 satisfies:

ii) $J(x, t, \phi)\in BUC(R^N\times[0, T])$ $(\forall T<\infty)$ (or in other words $S_0(\cdot)$ is strongly continuous),

*) Ceremade-Paris IX University, Place de Lattre de Tassigny. 75755 Paris Cedex 16 France.

**) Department of Mathematics, Kobe University, Rokko, Kobe 657 Japan.

iii) $S_0(t)\phi \leq S_0(t)\psi$, if $\phi \leq \psi$ in R^N, $(\forall t \geq 0)$,

iv) $\|S_0(t)\phi - S_0(t)\psi\|_{L^\infty(R^N)} \leq e^{-\lambda_0 t}\|\phi-\psi\|_{L^\infty(R^N)}$ where $\lambda_0 = \inf\{c(x, v)\,;\, x \in R^N, v \in V\}$, and

v) If ϕ, $D\phi$, $D^\alpha\phi \in X$ then $(1/t)(S^0(t)\phi-\phi) \xrightarrow[t\downarrow 0+]{} \mathcal{A}\phi$ in X,

uniformly for functions ϕ such that ϕ, $D\phi$, $D^2\phi$ are equicontinuous, where $\mathcal{A}$ stands for the *Hamilton-Jacobi-Bellman operator* defined on smooth functions by

$$\mathcal{A}\psi = \inf_{v \in V}\{A^v\psi(x) - f(x, v)\}, \qquad \text{for } \psi \in C_b^2(R^N)$$

and

$$A^v = \sum_{ij} a_{ij}(x, v)\partial_{ij} + \sum b_i(x, v)\partial_i - c(x, v), \qquad a(x, v) = \frac{1}{2}\sigma(x, v)\sigma^T(x, v).$$

Therefore, in some formal sense, $S_0(t)\phi(x) = u(x, t)$ is an integral solution of the following Cauchy problem for Hamilton-Jacobi-Bellman equation;

$$(6) \qquad \frac{\partial u}{\partial t} = \mathcal{A}u \qquad \text{in } R^N \times (0, \infty)$$

$$(7) \qquad u(x, 0) = \phi(x) \qquad \text{in } R^N.$$

In P. L. Lions [7], it is proved that if $\phi \in W^{2,\infty}(R^N)$, then u solves (6), (7) in some appropriate sense.

Recently, one of us (M. Nisio [12]) investigated the uniqueness of strongly continuous semigroup whose generator is an extension of the operator $\mathcal{A}$ (defined for example on $C_b^2(R^N)$). This question was solved under appropriate assumptions, using general results concerning abstract nonlinear semigroup theory (see P. Bénilan [1]) and the existence and uniqueness results on Hamilton-Jacobi-Bellman equations (HJB in short) proved by P. L. Lions [7].

We propose here a direct answer to that question by using the notion of *viscosity solution* of (6) introduced by M. G. Crandall and P. L. Lions (see P. L. Lions [8], [9]), extending the notion introduced in M. G. Crandall and P. L. Lions [4], [5] for first order Hamilton-Jacobi equations. We recall the definition of such solutions in § 2 below. The main feature of these solutions is shown by the following result proved in P. L. Lions [8] (see also [9]);

$S_0(t)\phi$ is the unique viscosity solution of (6), (7) in $BUC\,(R^N \times [0, T])$ $(\forall T < \infty)$.

This will enable us to show the following:

Theorem. *Let $(S(t))_{t\geq 0}$ be a strongly continuous semigroup on X satisfying*

$$(8) \qquad S(t)\phi \leq S(t)\psi \quad \textit{in } R^N, \quad \textit{if } \phi \leq \psi \quad \textit{in } R^N, \quad \forall t \geq 0$$

$$(9) \quad \forall\phi,\quad \psi \in \tilde{\mathcal{D}},\ (1/t)\{S(t)[\phi + t\psi] - \phi\}(x) \xrightarrow[t\downarrow 0+]{} \psi(x) + \mathcal{A}\phi(x), \quad \forall x \in R^N,$$

where $\tilde{\mathcal{D}} = \{\phi \in C^\infty(R^N),\ D^\alpha\phi \in C_b(R^N),\ \forall\alpha\}$. *Then,*

$$S(t)=S_0(t), \qquad \forall t \geq 0.$$

Remarks. i) The above result should be viewed as an example of application of the uniqueness of viscosity solution; in particular similar results will hold for controlled diffusion with boundary conditions, or for deterministic problems (differential games, etc.).

ii) It is clear that property (iv) of S_0 implies that S_0 satisfies (9).

iii) We do not know if the presence of ψ in (9) is necessary. By a simple modification of the proof below (remarking that by an approximation argument one can take below $\phi(x,t)$ of the form $\phi_1(x)+\phi_2(t)$) the result is still valid if we assume, instead of (9);

(9′) $\forall \phi \in \tilde{\mathcal{D}}, \ \forall r \in R, \ (1/t)\{S(t)(\phi+tr)-\phi\}(x) \xrightarrow[t\downarrow 0_+]{} r+\mathcal{A}\phi(x), \ \forall x \in R^N.$

In particular (9′) holds as soon as we have

$\lim_{t\downarrow 0+}(1/t)\{S(t)(\phi+tr)-S(t)\phi\}(x)=r, \ \forall \phi \in \tilde{\mathcal{D}}, \ \forall r \in R, \ \forall x \in R^N.$

2. Proof of Theorem. Let us first recall one possible form of the definition of viscosity solutions of (6) (see [3] [5]); $u \in C(R^N \times (0,\infty))$ is said to be a viscosity solution of (6), if for all $\phi \in C^2(R^N \times (0,\infty))$ then we have,

(10) $$\left(\frac{\partial \phi}{\partial t} - \inf_{v \in V}\left\{\sum_{ij} a_{ij}\partial_{ij}\phi + \sum_i b_i\partial_i\phi - cu + f\right\}\right) \leq 0,$$

at any local maximum (x_0, t_0) of $u-\phi$,

$$\left(\frac{\partial \phi}{\partial t} - \inf_{v \in V}\left\{\sum_{ij} a_{ij}\partial_{ij}\phi + \sum_i b_i\partial_i\phi - cu + f\right\}\right) \geq 0$$

at any local minimum (x_0, t_0) of $u-\phi$.

In addition it is enough to check (10) at any global extremum (x_0, t_0) of $u-\phi$ for $\phi \in \tilde{\mathcal{D}}(R^N \times [0,\infty))$ ($x_0 \in R^N$, $t_0>0$), see [3] [5] for related arguments. Therefore, by the uniqueness result recalled in the introduction, it is enough to check that $u(x,t)=(S(t)\phi)(x)$ is a viscosity solution of (6) and thus we will consider, for example, a global maximum point $(x_0,t_0) \in R^N \times (0,\infty)$ of $u-\phi$ where $\phi \in \tilde{\mathcal{D}}(R^N \times [0,\infty))$. Let $h \in (0, t_0)$, since without loss of generality we may assume $u(x_0,t_0)=\phi(x_0,t_0)$, we have by assumption (8)

$$\phi(x_0,t_0)=u(x_0,t_0)=\{S(h)u(t_0-h)\}(x_0) \leq \{S(h)\phi(t_0-h)\}(x_0)$$

where $u(t)(\cdot)=u(\cdot,t)$, $\phi(t)(\cdot)=\phi(\cdot,t)$. Now there exists $\varepsilon(h)>0$ for $h \in (0,t_0)$ such that

$$\begin{cases} \phi(x,t_0-h) \leq \phi(x,t_0) - h\dfrac{\partial \phi}{\partial t}(x,t_0) + h\varepsilon(h), & \text{in } R^N \\ \varepsilon(h) \longrightarrow 0 \quad \text{as} \quad h \longrightarrow 0. \end{cases}$$

Next, let $\varepsilon_0>0$, for $h \in (0, h_0(\varepsilon_0))$ we have $0<\varepsilon(h)<\varepsilon_0$. Thus using again (8), we deduce

$$\phi(x_0,t_0) \leq S(h)\{\phi(t_0)+h\psi\}(x_0)$$

with $\psi=-(\partial\phi/\partial t)(\cdot,t_0)+\varepsilon_0$. Using now assumption (9), we obtain dividing the above inequality by h and letting $h \to 0$,

$$\left(\frac{\partial \phi}{\partial t}-\mathcal{A}\phi\right)(x_0, t_0)-\varepsilon_0\leq 0.$$

We conclude, sending ε_0 to 0.

In the same way using the general uniqueness results of M. G. Crandall and P. L. Lions [4], [5], we have the following result on general first order Hamilton-Jacobi equations;

Corollary. Let $H\in C(R^N)$. Then there exists a unique strongly continuous semigroup $S_0(t)$ on X satisfying (8) and

(9″) $\forall\phi, \psi\in\tilde{\mathcal{D}}$, $(1/t)\{S_0(t)(\phi+t\psi)-\phi\}(x)\longrightarrow\psi(x)-H(D\phi(x))$, $\forall x\in R^N$,

and $(S_0(t)u_0)(x)$ is the unique viscosity solution in $BUC(R^N\times[0, T])$ (for all $T<\infty$) of

$$\begin{cases}\dfrac{\partial u}{\partial t}+H(Du)=0 & \text{in } R^N\times(0,\infty)\\ u(x,0)=u_0(x) & \text{in } R^N.\end{cases}$$

Remarks. i) Using the general results of [4], [5], we could consider as well general Hamiltonian $H(x, t, s, p)$.

ii) These uniqueness results for semigroup are used in P. L. Lions, G. Papanicolaou and S. R. S. Varadhan [10] in order to determine the limit of various asymptotic problems.

References

[1] P. Bénilan: Equation d'évolution dans un espace de Banach quelconque et applications. Thése d'Etat. Orsay (1972).

[2] A. Bensoussan and J. L. Lions: Applications des inéquations variationnelles en controle stochastique. Dunod (1978).

[3] M. G. Crandall, L. C. Evans, and P. L. Lions: Some properties of viscosity solutions of Hamiltonian-Jacobi equations (preprint).

[4] M. G. Crandall and P. L. Lions: Conditions d'unicité pour les solutions généralisées des équations de Hamilton-Jacobi du premier ordre. C. R. Paris, **252**, 183–186 (1981).

[5] ——: Viscosity solutions of Hamilton-Jacobi equations (to appear in Trans. Amer. Math. Soc.).

[6] N. V. Krylov: Controlled diffusion processes. Appl. Math., 14, Springer (1980) (translation from Russian).

[7] P. L. Lions: Control of diffusion processes in R^N. Comm. Pure Appl. Math., **34**, 121–147 (1981).

[8] ——: Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations (to appear).

[9] ——: Optimal stochastic control of diffusion type processes and Hamilton-Jacobi-Bellman equations. Proc. Cocoyoc Conf. Ed. W. H. Fleming and L. Gorostiza, Springer, Lect. Notes (to appear).

[10] P. L. Lions, G. Papanicolaou, and S. R. S. Varadhan: Work in preparation.

[11] M. Nisio: Stochastic Control Theory. ISI Lect. Notes 9, Macmillan, India (1981).

[12] ——: Note on uniqueness of semigroup associated with Bellman operator. Proc. Cocoyoc Conf. (ibid.).