71. *The Corona Problem on 2-Sheeted Disks*

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Consider the normed ring $H^\infty(R)$ of bounded analytic functions on a Riemann surface R equipped with the supremum norm. We denote by $M(R)$ the maximal ideal space of $H^\infty(R)$. We may identify each maximal ideal in $M(R)$ with a multiplicative functional χ on $H^\infty(R)$ with $\chi(1)=1$. An evaluation χ_p at $p \in R$ is a functional on $H^\infty(R)$ given by $\chi_p(f)=f(p)$. The natural map $\tau: R \to M(R)$ is defined by $\tau(p)=\chi_p$. The map τ is injective if and only if $H^\infty(R)$ separates the points in R and in this case we may identify $\tau(R)$ with R. We say that the corona theorem holds for R if $\tau(R)$ is dense in $M(R)$. Carleson [1] proved that the corona theorem holds for the 1-sheeted disk. We will prove the same for 2-sheeted disks. Namely

Theorem. *The corona theorem holds for any unbounded two sheeted covering surface of the unit disk.*

This provides us with nontrivial examples of Riemann surfaces of infinite genus for which the corona theorem holds. The proof will be given below in §§ 1–5.

1. Let (R, D, π) be an unbounded two sheeted covering surface of the unit disk $D=\{|z|<1\}$ with a covering map π so that $\pi^{-1}(z)$ consists of either two different points in R or a single point for each $z \in D$. If R has only a finite number of branch points, then R is conformally equivalent to an interior of a compact bordered Riemann surface for which the validity of the corona theorem is well known (cf. e.g. Gamelin [2]). Therefore we assume that the projection of branch points forms an infinite sequence $\{\alpha_n\}$ $(n=1,2,\cdots)$. Of course the proof given below is also valid for the case where the above sequence is finite.

2. Suppose first the case $\sum(1-|\alpha_n|)=+\infty$. Take an arbitrary $f \in H^\infty(R)$ and consider the $c \in H^\infty(D)$ defined by $c(z)=(f(z_1)-f(z_2))^2$ for $z \in D$ where $\pi^{-1}(z)=\{z_1, z_2\}$. Since $c(\alpha_n)=0$ $(n=1,2,\cdots)$, the Blaschke theorem assures that $c\equiv 0$ and a fortiori $H^\infty(R)=H^\infty(D)\circ\pi$. Therefore $M(R)=M(D)$ and $\tau(R)=D$, and the Carleson theorem [1] assures that $\bar{D}=M(D)$, i.e., $\overline{\tau(R)}=M(R)$.

3. We proceed to the case where $\sum(1-|\alpha_n|)<+\infty$. Let B be the Blaschke product with $\{\alpha_n\}$ its simple zero set. For simplicity we identify $H^\infty(D)\circ\pi$ with $H^\infty(D)$ so that we view $H^\infty(D)\subset H^\infty(R)$. For

any $f\in H^\infty(R)$ we consider c as in §2. By considering the Puiseux expansion of f at α_n we see that the order of zero of c at α_n is odd $(n=1,2,\cdots)$. The order of zero of c at any other zero than α_n is even. Therefore the orders of zeros of c/B are even. In view of this the Blaschke product with the zero set of c/B is of the form B_0^2 where B_0 is a Blaschke product. Since c/BB_0^2 is zero free on D, there exists a $d\in H^\infty(D)$ with $c/BB_0^2=d^2$. On setting $a=B_0d\in H^\infty(D)$, we have $f(z_1)-f(z_2)=a(z)\sqrt{B(z)}$. The function b given by $b(z)=f(z_1)+f(z_2)$ belongs to $H^\infty(D)$. We have the following representation:

(1) $$H^\infty(R)=\{a+b\sqrt{B}\ ;\ a,b\in H^\infty(D)\},$$

i.e. $H^\infty(R)$ is an $H^\infty(D)$-module generated by $\{1,\sqrt{B}\}$.

4. Since $\sqrt{B(z_1)}=-\sqrt{B(z_2)}$ for any $z\in D$ with $\pi^{-1}(z)=\{z_1,z_2\}$, (1) assures that $H^\infty(R)$ separates the points in R. Therefore we can identify $\tau(R)$ with R so that $R\subset M(R)$ similar to $D\subset M(D)$. We extend $\pi:R\to D$ to a mapping $\pi:M(R)\to M(D)$ by $\pi(p)(f)=p(f)$ for $(f,p)\in H^\infty(D)\times M(D)$. Then $\pi:M(R)\to M(D)$ is continuous. Given any $m\in M(D)$, we consider p_1 and p_2 in $M(R)$ given by

(2) $$\begin{cases} p_1(a+b\sqrt{B})=m(a)+m(b)\sqrt{m(B)},\\ p_2(a+b\sqrt{B})=m(a)-m(b)\sqrt{m(B)}.\end{cases}$$

It is easy to see that $\pi^{-1}(m)\supset\{p_1,p_2\}$. Take any $p\in\pi^{-1}(m)$. Observe that $p(\sqrt{B})^2=p(B)=m(B)$. Hence $p(a+b\sqrt{B})=p(a)+p(b)p(\sqrt{B})=m(a)+m(b)\sqrt{m(B)}$ or $m(a)-m(b)\sqrt{m(B)}$, i.e. $p=p_1$ or p_2. Thus we have

(3) $$\pi^{-1}(m)=\begin{cases}\{p_1,p_2\} & (m(B)\neq 0),\\ \{p_1=p_2\} & (m(B)=0).\end{cases}$$

5. Take any point $p\in M(R)$. Since $\pi(p)=m$ belongs to $\bar{D}$, there exists a net $(z_\alpha)\subset D$ converging to m. Set $\pi^{-1}(z_\alpha)=\{z_{1\alpha},z_{2\alpha}\}$. By (2) we see that

(4) $$z_{1\alpha}(\sqrt{B})=-z_{2\alpha}(\sqrt{B}).$$

Since $M(R)$ is compact, by choosing a subnet if necessary, we may assume that $(z_{j\alpha})$ is convergent to a point $p_j\in M(R)$ $(j=1,2)$. By the continuity of π, $\pi(p_j)=m$ $(j=1,2)$. By (3), $p\in\{p_1,p_2\}$ and $p\in\bar{R}$ since $p_j\in\bar{R}$. Thus the proof is complete.

References

[1] L. Carleson: Interpolations by bounded analytic functions and the corona problem. Ann. of Math., **76**, 547–559 (1962).

[2] T. Gamelin: Localization of the corona problem. Pacific J. Math., **34**, 74–81 (1970).