70. *Retraction and Extension of Mappings of M_1-Spaces*

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In this paper, we shall prove that an M_1-space X can be imbedded in an M_1-space $Z(X)$ as a closed subset in such a way that X is an AR ($\mathcal{M}_1$) (resp. ANR ($\mathcal{M}_1$)) if and only if X is a retract (resp. neighborhood retract) of $Z(X)$, where $\mathcal{M}_1$ is the class of all M_1-spaces. Moreover, we shall prove that an M_1-space is an AE ($\mathcal{M}_1$) (resp. ANE ($\mathcal{M}_1$)) if and only if it is an AR ($\mathcal{M}_1$) (resp. ANR ($\mathcal{M}_1$)).

Throughout this paper, all spaces are assumed to be Hausdorff topological spaces and all maps to be continuous. N denotes the set of all natural numbers. Let $\mathcal{C}$ be a class of spaces. For the definitions of AR ($\mathcal{C}$), ANR ($\mathcal{C}$), AE ($\mathcal{C}$) and ANE ($\mathcal{C}$), see [4]. Note that in [4] each class $\mathcal{C}$ is weakly hereditary; that is to say, if $\mathcal{C}$ contains X, then it contains every closed subspace of X. However, in this paper we consider the class $\mathcal{M}_1$ of all M_1-spaces though it is unknown if $\mathcal{M}_1$ is weakly hereditary.

1. Auxiliary lemma. For the definitions of uniformly approaching anti-cover and D-space, see [6]. The following lemma was essentially proved in the proof of [5, Lemma, 3.2].

Lemma 1.1. *Let X be a D-space, F a closed subset of X and f a map from F into a space Y. Let Y also denote the natural imbedding of Y in $X \bigcup_f Y = Z$. If $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ is a closure preserving open collection in Y, then for each $\alpha \in A$ there is a collection $\{U'_\beta : \beta \in B_\alpha\}$ of open subsets in Z satisfying the following three conditions:*

(C1) *$\mathcal{U}' = \{U'_\beta : \beta \in B_\alpha, \alpha \in A\}$ is closure preserving in Z,*

(C2) *for each $\beta \in B_\alpha$, $U'_\beta \cap Y = U_\alpha$, and for every open subset V in Z with $V \cap Y = U_\alpha$ there is $\beta \in B_\alpha$ such that $U_\alpha \subset U'_\beta \subset V$, and*

(C3) *for every open subset W in Y, there is an open subset W' of Z such that $W' \cap Y = W$ and $W' \cap U'_\beta = \phi$ whenever $\beta \in B_\alpha$ and $W \cap U_\alpha = \phi$.*

Proof. Let p be the projection from the free union $X \cup Y$ to Z. Since X is a D-space, X is an M_1-space. Therefore X is monotonically normal. Let G be a monotone normality operator for X satisfying the properties in [3, Lemma 2.2]. Since X is a D-space, F has a uniformly approaching anti-cover $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ in X. In particular, since X is hereditarily paracompact, we may assume that $\mathcal{V}$ is locally finite in $X - F$. For each $U_\alpha \in \mathcal{U}$, let $U'_\alpha = \cup\{G(x, F - p^{-1}(U_\alpha)) : x$

$\in p^{-1}(U_\alpha)\}$. Then U'_α is obviously open in X. For each $\alpha \in A$, let $B_\alpha = \{\gamma(\alpha) \subset \Lambda : p^{-1}(U'_{\gamma(\alpha)})$ is open in $U'_\alpha\}$, where $U'_{\gamma(\alpha)} = U_\alpha \cup p(\cup\{V_\lambda : \lambda \in \gamma(\alpha)\})$. Let $B = \cup\{B_\alpha : \alpha \in A\}$, and $\mathcal{U}' = \{U'_\beta : \beta \in B\}$. Then it is easy to see that the conditions (C1)–(C3) are satisfied by $\mathcal{U}'$.

2. Main theorems. In metric spaces, the closed imbedding theorem of Eilenberg-Wojdyslawski plays an important role in the development of retract theory. By using this theorem, it was shown that a metric space is an AE ($\mathcal{M}$) (resp. ANE ($\mathcal{M}$)) if and only if it is an AR ($\mathcal{M}$) (resp. ANR ($\mathcal{M}$)), where $\mathcal{M}$ is the class of all metric spaces. In [1], R. Cauty showed that a stratifiable space X can be imbedded in a stratifiable space $Z(X)$ as a closed subset in such a way that X is an AR ($\mathcal{S}$) (resp. ANR ($\mathcal{S}$)) if and only if X is a retract (resp. neighborhood retract) of $Z(X)$, where $\mathcal{S}$ is the class of all stratifiable spaces. In this section, for a space X we shall construct $Z(X)$ by using the method of R. Cauty [1], and prove the analogous results for the class of all M_1-spaces. For the definitions of M_1-space and stratifiable space, see [2].

Construction 2.1. Let X be a space. $M(X)$ denotes the full simplicial complex which has all points of X as the set of vertices. Then there is a canonical bijection i from the 0-skeleton M^0 of $M(X)$ onto X. Let $Z' = M(X) \bigcup_i X$ be the adjunction space and $p' : M(X) \cup X \to Z'$ the projection. By the aid of p', we identify X with $p'(X) \subset Z'$. Since the restriction of p' to $M(X)$ is a bijection from $M(X)$ onto Z', by the abuse of language, a simplex σ of $M(X)$ is said to be contained in a subset U of Z' if $p'(\sigma)$ is contained in U. $Z(X)$ denotes the space such that Z' is the underlying set of $Z(X)$ and the topology of $Z(X)$ has a base which consists of a collection of sets U, which is open in Z', satisfying the following condition:

(C) If σ is a simplex of $M(X)$ such that all vertices of σ are contained in $U \cap X$, then σ is contained in U.

Let $p : M(X) \cup X \to Z(X)$ be the projection. Then p is obviously continuous. Let M^n be the n-skeleton of $M(X)$ and $Z^n = p(M^n \cup X)$.

Lemma 2.2. *If X is an M_1-space, then $Z(X)$ is also M_1.*

Proof. For each $n \in N$, let Y be the free union of all $(n+1)$-simplexes of $M(X)$, F the boundary of Y and $f : F \to Z^n$ the map defined by $f(x) = p(x)$ for $x \in F$. Then the set $Y \bigcup_f Z^n$ is equal to the set Z^{n+1}. Let $\{U_\alpha : \alpha \in A\}$ be a closure preserving open collection in Z^n. Since Y is a metric space, Y is a D-space. Therefore the technique of proof of Lemma 1.1 yields that, for each $\alpha \in A$, there is a collection $\{U'_\beta : \beta \in B_\alpha\}$ of open subsets in Z^{n+1} satisfying (C1)–(C3). (Note that this proof is slightly different from that of Lemma 1.1; i.e. if σ is (n+1)-simplex and U_α contains all vertices of σ, then σ is contained in U'_β, $\beta \in B_\alpha$.)

Now, let $\{U(\alpha_1) : \alpha_1 \in A\}$ be a closure preserving open collection in X $(=Z^0)$. From the preceding paragraph we get that every $U(\alpha_1)$ can be extended to open sets $\{U(\alpha_1, \alpha_2) : \alpha_2 \in A(\alpha_1)\}$ in Z^1 in such a way that the collection $\{U(\alpha_1, \alpha_2) : \alpha_1 \in A, \alpha_2 \in A(\alpha_1)\}$ satisfies (C1)–(C3). Repeating this process, we get for each $n \in N$ a closure preserving open collection $\{U(\alpha_1, \cdots, \alpha_{n+1}) : \alpha_1 \in A, \ \alpha_2 \in A(\alpha_1), \cdots, \alpha_{n+1} \in A(\alpha_1, \cdots, \alpha_n)\}$ in Z^n. Let $\Sigma = \{(\alpha_1, \alpha_2, \alpha_3, \cdots) : \alpha_1 \in A, \ \alpha_2 \in A(\alpha_1), \ \alpha_3 \in A(\alpha_1, \alpha_2), \cdots\}$. For each $(\alpha_1, \alpha_2, \cdots) \in \Sigma$, let $U(\alpha_1, \alpha_2, \cdots) = \cup \{U(\alpha_1, \cdots, \alpha_n) : n \in N\}$. Then it is easy to see that $U(\alpha_1, \alpha_2, \cdots)$ is open in $Z(X)$ and $\mathcal{U} = \{U(\alpha_1, \alpha_2, \cdots) : (\alpha_1, \alpha_2, \cdots) \in \Sigma\}$ is closure preserving in $Z(X)$.

Finally, let $\{\mathcal{U}_n\}$ is a σ-closure preserving base for X. Then it is easily verified that the extensions $\{\mathcal{U}'_n\}$ of $\{\mathcal{U}_n\}$ to $Z(X)$, by the same method above, is a σ-closure preserving base at each point of X. Furthermore, since $M(X)$ is an M_1-space by [2, Theorem 8.3] and the open subspace $Z(X)-X$ is homeomorphic to an open subspace $M(X)$, there exists a σ-closure preserving base $\{\mathcal{V}_n\}$ at each point of $Z(X)-X$. Thus $\{\mathcal{U}'_n\} \cup \{\mathcal{V}_n\}$ is a σ-closure preserving base for $Z(X)$. This completes the proof.

The following lemma was proved in [1, Lemma 1.2].

Lemma 2.3. *Let X be a space. If Y is a stratifiable space, A a closed subset of Y and $f : A \to X$ a map, then there is a map $F : Y \to Z(X)$ with $F|A = f$.*

The following theorem is an immediate consequence of Lemmas 2.2 and 2.3.

Theorem 2.4. *An M_1-space X is an* AR $(\mathcal{M}_1)$ (resp. ANR $(\mathcal{M}_1)$) *if and only if X is a retract* (resp. neighborhood retract) *of $Z(X)$.*

The following theorem is a direct consequence of Theorem 2.4 and Lemma 2.3. Note that whether the class $\mathcal{M}_1$ is weakly hereditary is a long-standing unsolved question first posed by Ceder [2].

Theorem 2.5. *An M_1-space is an* AE $(\mathcal{M}_1)$ (resp. ANE $(\mathcal{M}_1)$) *if and only if it is an* AR $(\mathcal{M}_1)$ (resp. ANR $(\mathcal{M}_1)$).

References

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