69. On Ξ-Product of Spaces which have a σ-Almost Locally Finite Base

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1. Introduction. Let $\{X_a : a \in A\}$ be a family of topological spaces. By $B_a X_a$ we denote the set $\prod_a X_a$ with the box product topology. For $p \in B_a X_a$ we denote the subspace $\{x \in B_a X_a : x_a \neq p_a$ for at most finitely many $a\}$ of $B_a X_a$ by Ξ_p.

Recently K. Tamano and the author [3] introduced the notion of almost local finiteness and the class of all spaces which have a σ-almost locally finite base. This class is an intermediate class between that of free L-spaces and that of M_1-spaces. The purpose of this paper is to prove that Ξ_p has a σ-almost locally finite base if each X_a has a σ-almost locally finite base and $p \in B_a X_a$. Corollary 3.2 is an improvement on the result of S. San-ou [5]. By [4], Ξ_p need not be free L even if each X_a is metrizable and $p \in B_a X_a$. For another results on Ξ-product see [1], [2] and [5].

In this paper all spaces are assumed to be regular T_1.

2. Preliminaries. Definition 2.1. Let X be a space and $\mathcal{A}$ a family of subsets of X. $\mathcal{A}$ is said to be *almost locally finite* in X if for every point x of X there exist a neighborhood U of x and a finite family $\mathcal{B}$ of subsets of X such that

$\{A \cap U : A \in \mathcal{A}\} \subset \{B \cap W : B \in \mathcal{B}$ and W is a neighborhood of $x\}$.

Lemma 2.2. *Let $\{X_e : e \in E\}$ be a family of spaces and $p \in B_e X_e$. For each $e \in E$ let $\mathcal{A}_e$ be an almost locally finite family of open sets of X_e such that*

$$\text{if } V \in \mathcal{A}_e \text{ then } p_e \in V \text{ or } p_e \notin \operatorname{Cl} V.$$

Then $\{\Xi_p \cap \prod_e V_e : (V_e)_{e \in E} \in \prod_e \mathcal{A}_e\}$ is almost locally finite in Ξ_p.

Proof. Let $x \in \Xi_p$.

Case 1. $x = p$.

For each $e \in E$ put $U_e = X_e - \cup\{\operatorname{Cl} V : V \in \mathcal{A}_e, p_e \notin \operatorname{Cl} V\}$. Put $U = \Xi_p \cap \prod_e U_e$. Then U is a neighborhood of x. Let $(V_e)_{e \in E} \in \prod_e \mathcal{A}_e$ and $U \cap \prod_e V_e \neq \emptyset$. Then $x_e = p_e \in V_e$, $e \in E$. Therefore $U \cap \prod_e V_e$ is a neighborhood of x.

Case 2. $x \neq p$.

Let $E_1 = \{e \in E : x_e = p_e\}$ and $E_2 = E - E_1$. Then $|E_2| < \aleph_0$. For $e \in E_1$ put $U_e = X_e - \cup\{\operatorname{Cl} V : V \in \mathcal{A}_e, p_e \notin \operatorname{Cl} V\}$. For $e \in E_2$ there exist a neighborhood U_e of x_e and a finite family $\mathcal{B}_e$ of subsets of X_e such that

$$\{V\cap U_e : V\in\mathcal{A}_e\}\subset\{B\cap W : B\in\mathcal{B}_e,\ W \text{ is a neighborhood of } x_e\}.$$

Put $U=\Xi_p\cap\prod_e U_e$ and

$$\mathcal{B}=\{\Xi_p\cap(\prod_{e\in E_2}B_e\times\prod_{e\in E_1}X_e) : (B_e)_{e\in E_2}\in\prod_{e\in E_2}\mathcal{B}_e\}.$$

Then U is a neighborhood of x and $\mathcal{B}$ is a finite family of subsets of Ξ_p. Let $(V_e)_{e\in E}\in\prod_e\mathcal{A}_e$ and $U\cap\prod_e V_e\neq\emptyset$. Then $x_e\in V_e$, $e\in E_1$. For $e\in E_2$ there exist $B_e\in\mathcal{B}_e$ and a neighborhood W_e of x_e such that $V_e\cap U_e=B_e\cap W_e$. Then

$$U\cap\prod_e V_e=\Xi_p\cap(\prod_{e\in E_1}(U_e\cap V_e)\times\prod_{e\in E_2}W_e)\cap(\prod_{e\in E_1}X_e\times\prod_{e\in E_2}B_e);$$

$\Xi_p\cap(\prod_{e\in E_1}(U_e\cap V_e)\times\prod_{e\in E_2}W_e)$ is a neighborhood of x; and

$\Xi_p\cap(\prod_{e\in E_1}X_e\times\prod_{e\in E_2}B_e)\in\mathcal{B}$.

Therefore $\{\Xi_p\cap\prod_e V_e : (V_e)_{e\in E}\in\prod_e\mathcal{A}_e\}$ is almost locally finite in Ξ_p.

3. The theorem. **Theorem 3.1.** *Let $\{X_e : e\in E\}$ be a family of spaces which have a σ-almost locally finite base and $p\in B_eX_e$. Then Ξ_p has a σ-almost locally finite base.*

Proof. Obviously Ξ_p is regular T_1. For each $e\in E$ let $\cup\{\mathcal{B}_n^e : n\in N\}$ be a σ-almost locally finite base of X_e such that

$$\mathcal{B}_n^e \text{ is almost locally finite, } n\in N; \quad\text{and}$$
$$\mathcal{B}_n^e\subset\mathcal{B}_{n+1}^e,\ n\in N.$$

By Theorem 3.4 of [3], p_e has an almost locally finite open neighborhood base $\mathcal{O}(p_e)$, $e\in E$. For each $e\in E$, take a family $\{G_n^e : n\in N\}$ of open sets of X_e such that

$$\mathrm{Cl}\,G_n^e\subset G_{n+1}^e,\ n\in N;\quad\text{and}$$
$$X_e-\{p_e\}=\cup\{G_n^e : n\in N\}.$$

Put $\mathcal{O}_n^e=\mathcal{O}(p_e)\cup\{V\cap G_n^e : V\in\mathcal{B}_n^e\}$, then by Propositions 2.6 and 2.8 of [3], $\mathcal{O}_n^e$ is almost locally finite and satisfies the condition of Lemma 2.2. Let

$$\mathcal{B}_n=\{\Xi_p\cap\prod_e V_e : (V_e)_{e\in E}\in\prod_e\mathcal{O}_n^e\},\qquad n\in N.$$

Then by Lemma 2.2, each $\mathcal{B}_n$ is almost locally finite. It is easy to show that $\cup\{\mathcal{B}_n : n\in N\}$ is a base of Ξ_p. Thus the proof is completed.

Corollary 3.2. *Let $\{X_e : e\in E\}$ be a family of metric spaces, $p\in B_eX_e$ and $X\subset\Xi_p$. Then*

(1) *X is an M_1-space; and*

(2) *every closed image of X is an M_1-space.*

Proof. These follow from Theorem 3.1 and Theorems 3.2, 3.3 and 3.6 of [3].

References

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