67. *On a Certain Inverse Problem for the Heat Equation on the Circle**)

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In the previous work [3]–[5], the author considered the heat equation $u_t=u_{xx}-p(x)u$ $(0<t<\infty,\ 0<x<1)$ on the compact interval $[0,1]$ with the boundary condition $u_x-hu|_{x=0}=u_x+Hu|_{x=1}=0$ $(0<t<\infty)$ and with the initial condition $u|_{t=0}=a(x)$ $(0<x<1)$, which is denoted by $(E_{p,h,H,a})$, and studied the problem to determine the coefficients p, h, H and the initial value a from the values of the solution on the boundary or on some interior point $x_0\in(0,1)$, and so on. In the present paper, we consider the same equation on the circle S^1, the compact interval $[0,1]$ with end points identified, and study similar problems as those.

Namely, for $p\in C^1(S^1)$ and $a\in L^2(S^1)$, let $(E^s_{p,a})$ denote the heat equation

(1) $$\frac{\partial u}{\partial t}+\left(p(x)-\frac{\partial^2}{\partial x^2}\right)u=0 \qquad (0<t<\infty,\ x\in S^1)$$

with the initial condition

(2) $$u|_{t=0}=a(x) \qquad (x\in S^1),$$

and let A^s_p be the realization in $L^2(S^1)$ of the differential operator $p(x)-\partial^2/\partial x^2$. Henceforth $p\in C^1(S^1)$ means $p\in C^1(\mathcal{R}^1)$ and $p(x+1)=p(x)$. The problem which we have in mind here is, for instance, as follows: Do the values $\{u(t,x_0),\ u_x(t,x_0)\,|\,T_1\leqq t\leqq T_2\}$ determine uniquely (p,a), where $x_0\in S^1$ and $0\leqq T_1<T_2<\infty$? However, this question is negative without any assumption on (p,a). For example, $u\equiv 0$ holds for each p if $a\equiv 0$. In fact, for the equation $(E_{p,h,H,a})$, Suzuki [5] showed that the values $\{u(t,1/2),\ u_x(t,1/2)\,|\,T_1\leqq t\leqq T_2\}$ determine uniquely (p,h,H,a) if a is "a generating element with respect to $A_{p,h,H}$", where $A_{p,h,H}$ denotes the realization in $L^2(0,1)$ of the differential operator $p(x)-\partial^2/\partial x^2$ with the boundary condition, and where $a\in L^2(0,1)$ is said to be a generating element with respect to $A_{p,h,H}$ iff it is not orthogonal to any eigenfunction of $A_{p,h,H}$. Similar results are obtained by Suzuki [3], [4] for other inverse problems for $(E_{p,h,H,a})$, and also by Suzuki-Murayama [7], Murayama [1] and the papers referred by them. See Suzuki [6], for these works.

In order to generalize the notion of "generating" to our problem,

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however, we have to pay attention to the multiplicities of eigenvalues of A_p^s. We recall that each eigenvalue of $A_{p,h,H}$ is simple, while that of A_p^s may be double. Indeed, S. Nakagiri considered in [2] a general parabolic equation in a bounded domain $\Omega\subset\mathcal{R}^N$ with smooth coefficients, and studied the problem to determine the coefficients of it from full informations of several solutions: $\{u^j(t,x)\,|\,0\leqq t<\infty,\ x\in\bar{\Omega},\ 1\leqq j\leqq\alpha\}$. He showed that if the set of initial values $\{a^j\equiv u^j|_{t=0}\,|\,1\leqq j\leqq\alpha\}$ satisfies the so-called "rank condition", then $\{u^j(t,x)\,|\,0\leqq t<\infty,\ x\in\bar{\Omega},\ 1\leqq j\leqq\alpha\}$ determine uniquely the coefficients. We now recall it and introduce the notion of "a generating set of initial values" according to [2].

Notation. The eigenvalues of A_p^s are denoted by $\{\lambda_n\}_{n=0}^\infty$ $(-\infty<\lambda_0<\lambda_1<\cdots\to\infty)$. For each $n=0,1,2,\cdots$, the multiplicity of λ_n is denoted by $\alpha(n)$, and $\{\phi_{nl}\,|\,1\leqq l\leqq\alpha(n)\}$ denotes the set of eigenfunctions of A_p^s corresponding to λ_n, normalized by $\|\phi_{nl}\|_{L^2(S^1)}=1$.

We note $\alpha(n)=1$ or 2 $(n=0,1,2,\cdots)$ and set $\alpha\equiv\max_n\alpha(n)$.

Definition. A set of initial values $\{a^j\,|\,1\leqq j\leqq\alpha\}$ is said to be generating with respect to A_p^s, iff the matrix

$$A_n=((a^j,\phi_{nl})_{L^2(S^1)})_{1\leqq j\leqq\alpha,\ 1\leqq l\leqq\alpha(n)} \tag{3}$$

satisfies the following rank condition for each n:

$$\operatorname{rank} A_n=\alpha(n)\qquad(n=0,1,2,\cdots). \tag{4}$$

Here $(\ ,\ )_{L^2(S^1)}$ means the inner product in $L^2(S^1)$.

Now we state our results. To this end, assume that $\{a^j\,|\,1\leqq j\leqq\alpha\}$ is a generating set of initial values with respect to A_p^s, and let $u^j=u^j(t,x)$ denote the solution of (E_{p,a^j}) $(1\leqq j\leqq\alpha)$. Furthermore, let $v^j=v^j(t,x)$ denote other solutions of (E^s_{q,b^j}) for some $q\in C^1(S^1)$ and $b^j\in L^2(S^1)$ $(1\leqq j\leqq\alpha)$, and let T_1,T_2 in $0\leqq T_1<T_2<\infty$ be given. Then,

Theorem 1. *If $x_1\in S^1$ and $x_2\in S^1$ satisfy the central symmetry, say $x_1=1/2$ and $x_2=1$ $(=0)$, then the equalities*

$$\begin{aligned} &v^j(t,x_1)=u^j(t,x_1), && v^j_x(t,x_1)=u^j_x(t,x_1)\\ &v^j(t,x_2)=u^j(t,x_2) && (T_1\leqq t\leqq T_2,\ 1\leqq j\leqq\alpha)\end{aligned} \tag{5}$$

imply

$$(q,b^j)=(p,a^j)\qquad(1\leqq j\leqq\alpha). \tag{6}$$

Theorem 2. *Suppose that $x_1\in S^1$ and $x_2\in S^1$ don't satisfy the central symmetry, and let $x_1'\in S^1$ and $x_2'\in S^1$ be the symmetric point of x_1 and x_2, respectively, say $x_1=1/2$, $1/2<x_2<1$, $x_1'=1$ $(=0)$ and $x_2'=x_2-1/2$. Let A, A', B and B' be the arcs $\widehat{x_1x_2}$, $\widehat{x_1'x_2'}$, $\widehat{x_1'x_2}$ and $\widehat{x_1x_2'}$, respectively, as in Fig. 1. Then, the equalities* (5) *imply*

$$q(x)=p(x)\qquad(x\in A\cup A'). \tag{7}$$

Theorem 3. *Under the same circumstances as those of Theorem 2, the equalities* (5) *combined with either $q(x)=p(x)$ $(x\in B)$ or $q(x)=p(x)$ $(x\in B')$ imply* (6).

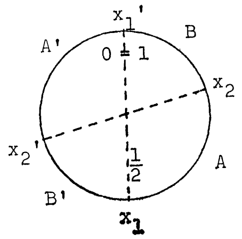

Fig. 1

Theorem 4. *In the case of* $p(x+1/2)=p(x)$ *and* $q(x+1/2)=q(x)$ $(x \in \mathcal{R}^1)$,

$$(5') \qquad v^j(t, x_1)=u^j(t, x_1), \qquad v^j_x(t, x_1)=u^j_x(t, x_1)$$
$$(T_1 \leqq t \leqq T_2,\ 1 \leqq j \leqq \alpha)$$

imply (6), *where* $x_1 \in S^1$.

Similar theorems are obtained for inverse spectral problems for the Hill equation. Details will be published elesewhere along with the proof of Theorems 1–4, which is based on lemmas by [5].

References

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