65. *On Invariant Differential Operators on Bounded Symmetric Domains of Type IV*

By Shōichi NAKAJIMA
Department of Mathematics, University of Tokyo

(Communicated by Kunihiko KODAIRA, M. J. A., June 15, 1982)

We shall give an explicit calculation of a system of generators of the algebra of invariant differential operators on a bounded symmetric domain of type IV.

For each $q \geq 3$ ($q \in \boldsymbol{N}$), put

$$\mathfrak{D}_q = \left\{ z = {}^t(z_1, \cdots, z_q) \in \boldsymbol{C}^q \mid y_1 y_q - \sum_{i=2}^{q-1} y_i^2 > 0,\ y_1 > 0 \right\},$$

where $z_i = x_i + \sqrt{-1}\, y_i$, $x_i, y_i \in \boldsymbol{R}$ $(1 \leq i \leq q)$. Then $\mathfrak{D}_q$ is a Hermitian symmetric space of rank 2, holomorphically isomorphic to a bounded symmetric domain of type IV. The group $SO_o(2, q)$ acts on $\mathfrak{D}_q$, and $\mathfrak{D}_q \cong SO_o(2, q)/SO(2) \times SO(q)$ as homogeneous spaces.

For $q=3$, $\mathfrak{D}_3$ is also isomorphic to $\mathfrak{H}_2 = \{Z \in M_2(\boldsymbol{C}) \mid {}^tZ = Z,\ \mathrm{Im}\, Z > 0\}$, the Siegel upper half-plane of genus 2. An isomorphism $\mathfrak{D}_3 \to \mathfrak{H}_2$ is defined by ${}^t(z_1, z_2, z_3) \to \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}$.

Let $\boldsymbol{D}(\mathfrak{D}_q)$ be the $\boldsymbol{C}$-algebra of all $SO_o(2, q)$-invariant differential operators on $\mathfrak{D}_q$. Since $\mathfrak{D}_q$ is of rank 2, $\boldsymbol{D}(\mathfrak{D}_q)$ is isomorphic to the polynomial ring of 2 variables over $\boldsymbol{C}$. A system of generators of $\boldsymbol{D}(\mathfrak{D}_q)$ will be given explicitly in Theorem.

The author would like to express his hearty thanks to Profs. T. Ibukiyama and Takayuki Oda, who gave him warm encouragements and helpful advices related to this subject.

Let $\mathfrak{D}_q$ be as above, with the following action of $SO_o(2, q)$: Let Q_o and Q be the following symmetric matrices:

$$Q_o = \left(\begin{array}{c|c|c} 0 & 0 & \frac{1}{2} \\ \hline 0 & -E_{q-2} & 0 \\ \hline \frac{1}{2} & 0 & 0 \end{array}\right) \in M_q(\boldsymbol{R}),$$

$$Q = \left(\begin{array}{c|c|c} 0 & 0 & \frac{1}{2} \\ \hline 0 & Q_o & 0 \\ \hline \frac{1}{2} & 0 & 0 \end{array}\right) \in M_{q+2}(\boldsymbol{R}),$$

where E_{q-2} is the identity matrix of degree $q-2$. Put

$O(Q_o)=\{g \in GL_q(\boldsymbol{R}) \mid {}^t g Q_o g = Q_o\}$,

$SO(Q_o)=\{g \in O(Q_o) \mid \det(g)=1\}$,

$SO_o(Q_o)=$ the connected component of the identity of $SO(Q_o)$,

and define $O(Q)$, $SO(Q)$ and $SO_o(Q)$ similarly using Q instead of Q_o. Then $SO_o(Q)$ is isomorphic to $SO_o(2, q)$, and acts on $\mathfrak{D}_q$ in the following way. For $z={}^t(z_1, \cdots, z_q) \in \mathfrak{D}_q$, put

$$\tau(z)={}^t\left(-z_1z_q+\sum_{i=2}^{q-1} z_i^2,\ -z_1, \cdots, -z_q, 1\right) \in \boldsymbol{C}^{q+2}.$$

Then the action of $g \in SO_o(Q)$ on z, $g: z \to g\cdot z$, is defined by

$$g\cdot\tau(z)=\varepsilon\cdot\tau(g\cdot z) \qquad \varepsilon=\varepsilon(g, z) \in \boldsymbol{C}^\times,$$

where, on the left side, the action of g is the linear one ($g \in GL_{q+2}(\boldsymbol{C})$).

Through this action of $SO_o(Q) \cong SO_o(2, q)$, $\mathfrak{D}_q$ can be identified with $SO_o(2, q)/SO(2) \times SO(q)$. For a proof, see [1] chap. 6, § 3. (In [1], the coefficients are slightly different from ours.)

For a function f on $\mathfrak{D}_q$ and $g \in SO_o(Q)$, we define f^g by the equation $f^g(z)=f(g\cdot z)$, for $z \in \mathfrak{D}_q$. If X is a differential operator on $\mathfrak{D}_q$, we define X^g by $X^g(f)=[X(f^{g^{-1}})]^g$ for any function f on $\mathfrak{D}_q$. Finally, we define $\boldsymbol{D}(\mathfrak{D}_q)$ as the $\boldsymbol{C}$-algebra of all differential operators X on $\mathfrak{D}_q$ such that $X^g=X$ for all $g \in SO_o(Q)$.

In order to write down the differential operators on $\mathfrak{D}_q$, we use the following abbreviations.

$$\begin{aligned} \partial_i &= \frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right) \\ \bar\partial_i &= \frac{\partial}{\partial \bar z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right) \end{aligned} \qquad (1 \le i \le q)$$

$\sum$ [resp. $\sum'$] will always mean $\sum_{i=1}^{q}$ $\left[\text{resp. } \sum_{i=2}^{q-1}\right]$.

Theorem. *Let Δ_1 and Δ_2 be the following differential operators on $\mathfrak{D}_q$.*

$$\Delta_1=\sum_{i,j=1}^{q} y_i y_j \partial_i \bar\partial_j - d(\partial_1\bar\partial_q + \bar\partial_1\partial_q - \textstyle\sum' \partial_i\bar\partial_i),$$

$$\begin{aligned} \Delta_2 = {} & d^2\left(\partial_1\partial_q - \frac{1}{4}\textstyle\sum'\partial_i^2\right)\left(\bar\partial_1\bar\partial_q - \frac{1}{4}\textstyle\sum'\bar\partial_i^2\right) \\ & + \sqrt{-1}\frac{q-2}{4} d(\textstyle\sum y_i\partial_i)\left(\bar\partial_1\bar\partial_q - \frac{1}{4}\textstyle\sum'\bar\partial_i^2\right) \\ & - \sqrt{-1}\frac{q-2}{4} d(\textstyle\sum y_i\bar\partial_i)\left(\partial_1\partial_q - \frac{1}{4}\textstyle\sum'\partial_i^2\right) \\ & + \frac{(q-2)^2}{16} d\left(\partial_1\bar\partial_q + \bar\partial_1\partial_q - \frac{1}{2}\textstyle\sum'\partial_i\bar\partial_i\right), \end{aligned}$$

where $d=y_1y_q-\sum' y_i^2$.

Then they belong to $\boldsymbol{D}(\mathfrak{D}_q)$, and $\boldsymbol{D}(\mathfrak{D}_q)$ is the polynomial ring of 2 *variables generated by Δ_1 and Δ_2. The operator Δ_1 is, up to a constant multiple, the Laplace operator of $\mathfrak{D}_q$.*

Proof. By using Th. 6.15 in [2] Chap. X, we can show the following facts. The algebra $\boldsymbol{D}(\mathfrak{D}_q)$ is isomorphic to the polynomial ring of 2 variables and its 2 generators are necessarily of ranks 2 and 4. If Δ_1' and Δ_2' generate $\boldsymbol{D}(\mathfrak{D}_q)$ (rank of $\Delta_i'=2i$, $i=1,2$), the elements of rank 2 [resp. of rank 4] in $\boldsymbol{D}(\mathfrak{D}_q)$ are of the form $a\Delta_1'$, $a\in \boldsymbol{C}^\times$ [resp. $b\Delta_2'+c(\Delta_1')^2+d\Delta_1'+e$; $b,\cdots,e\in\boldsymbol{C}$, b or $c\neq 0$]. Hence, Δ_1' is, up to a constant multiple, the Laplace operator of $\mathfrak{D}_q$.

Suppose we could prove that Δ_1 and Δ_2 belong to $\boldsymbol{D}(\mathfrak{D}_q)$. Since Δ_1 and Δ_2 are of ranks 2 and 4, they are of the form $\Delta_1=a\Delta_1'$, $\Delta_2=b\Delta_2'+c(\Delta_1')^2+d\Delta_1'+e$; $a,\cdots,e\in\boldsymbol{C}$, $a\neq 0$, b or $c\neq 0$. Here, we can see $b\neq 0$, because it is easily verified that $\Delta_2\notin \boldsymbol{C}[\Delta_1]$. This means that Δ_1 and Δ_2 also generate $\boldsymbol{D}(\mathfrak{D}_q)$, and $\boldsymbol{D}(\mathfrak{D}_q)$ is the polynomial ring generated by Δ_1 and Δ_2.

So, we shall prove that Δ_1 and Δ_2 belong to $\boldsymbol{D}(\mathfrak{D}_q)$, i.e. that they are invariant by $SO_o(Q)$.

Let p_ξ, $k_{a,A}$ and w be the following elements of $SO_o(Q)$.

$$p_\xi=\left(\begin{array}{c|c|c} 1 & \tilde{\xi} & \mu \\ \hline 0 & E_q & -\xi \\ \hline 0 & 0 & 1 \end{array}\right),\qquad \xi={}^t(\xi_1,\cdots,\xi_q)\in\boldsymbol{R}^q,$$

where $\tilde{\xi}=(\xi_q,-2\xi_2,\cdots,-2\xi_{q-1},\xi_1)$, and $\mu=-\xi_1\xi_q+\sum'\xi_i^2$,

$$k_{a,A}=\left(\begin{array}{c|c|c} a & 0 & 0 \\ \hline 0 & A & 0 \\ \hline 0 & 0 & a^{-1} \end{array}\right),\quad a\in\boldsymbol{R}^\times,\quad A\in O(Q_o),$$

$$w=\left[\begin{array}{c|c|c} 0 & 0 & \begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix} \\ \hline 0 & E_{q-2} & 0 \\ \hline \begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix} & 0 & 0 \end{array}\right].$$

They act on $\mathfrak{D}_q$ as follows;

$$p_\xi\cdot z=z+\xi,$$
$$k_{a,A}\cdot z=aAz,$$
$$w\cdot z=\frac{1}{\delta}\,{}^t(-z_q,z_2,\cdots,z_{q-1},-z_1),$$

where $\delta=z_1z_q-\sum' z_i^2$.

We can show that p_ξ, $k_{a,A}$ and w ($\xi\in\boldsymbol{R}^q$, $a\in\boldsymbol{R}^\times$, $A\in O(Q_o)$) generate $SO_o(Q)$. Therefore, to prove that Δ_1 and Δ_2 are invariant by $SO_o(Q)$, it is sufficient to verify that they are invariant by p_ξ, $k_{a,A}$ and w.

(I) Invariance by p_ξ and $k_{a,A}$. Obviously, Δ_1 and Δ_2 are invariant by p_ξ, and we can easily verify that they are invariant by $k_{a,A}$, if we notice the equation $\partial_1\partial_q-(1/4)\sum'\partial_i^2=(1/4)(\partial_1,\cdots,\partial_q)Q_o^{-1t}(\partial_1,\cdots,\partial_q)$.

(II) Invariance by w. The action of w is as follows;

$$\begin{aligned}
&\delta^w=\delta^{-1} && (\delta=z_1z_q-\textstyle\sum' z_i^2),\\
&d^w=(\delta\bar{\delta})^{-1}d && (d=y_1y_q-\textstyle\sum' y_i^2),\\
&\partial_1^w=-\delta\partial_q+z_1D, &&\\
&\partial_q^w=-\delta\partial_1+z_qD, &&\\
&\partial_i^w=\delta\partial_i+2z_iD && (2\le i\le q-1),
\end{aligned}$$

where $D=\sum z_i\partial_i$. Since the action of w is commutative with complex conjugation, we can see the action of w on $\bar{\partial}_i$'s. After straightforward computations, we get

$$\left(\partial_1\partial_q-\frac{1}{4}\sum'\partial_i^2\right)^w=\delta^2\left(\partial_1\partial_q-\frac{1}{4}\sum'\partial_i^2\right)-\frac{q-2}{2}\delta D,$$

and furthermore, if we put $D^*=\sum y_i\partial_i$, we have

$$(D^*)^w=\delta(\bar{\delta})^{-1}D^*-2\sqrt{-1}\,d(\bar{\delta})^{-1}D,$$

and

$$\left(\partial_1\bar{\partial}_q+\bar{\partial}_1\partial_q-\frac{1}{2}\sum'\partial_i\bar{\partial}_i\right)^w=\delta\bar{\delta}\left(\partial_1\bar{\partial}_q+\bar{\partial}_1\partial_q-\frac{1}{2}\sum'\partial_i\bar{\partial}_i\right)$$
$$+2\sqrt{-1}\,\delta D^*\bar{D}-2\sqrt{-1}\,\bar{\delta}\overline{D^*}D+4dD\bar{D}.$$

If we rewrite Δ_1^w and Δ_2^w by using these equations, and if we rearrange them, we obtain $\Delta_1^w=\Delta_1$ and $\Delta_2^w=\Delta_2$.

By (I) and (II), we know that Δ_1 and Δ_2 belong to $\boldsymbol{D}(\mathfrak{D}_q)$. Therefore, as stated at the beginning of the proof, all assertions of Theorem are proved.

References

[1] W. L. Baily, Jr.: Introductory Lectures on Automorphic Forms. Publ. Math. Soc. Japan, 12 (1973).

[2] S. Helgason: Differential Geometry and Symmetric Spaces. Academic Press (1962).