64. *Logarithmic Deformations of Holomorphic Maps and Equisingular Displacements of Surfaces with Ordinary Singularities**)

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Introduction. In this paper we shall give a definition of logarithmic deformations of holomorphic maps and prove the existence of the semi-universal family. If the map we consider is non-degenerate, this family turns out to be universal. The concept of logarithmic deformations of holomorphic maps is stimulated by Y. Kawamata's paper [3].

As a by-product we shall obtain another proof of the existence of the universal family of equisingular displacements of surfaces with ordinary singularities, which has been already proved by M. Namba [6]. Following Y. Kawamata, we use the terminology "equisingular" in the sense that they admit a simultaneous *embedded* resolution. Our result cannot cover K. Kodaira's existence theorem [4]. However, in case that the ambient threefold W satisfies the condition $H^2(W, \mathcal{O}_W)=0$, our theorem includes it.

Our method is expected to be useful for the proof of the higher dimensional analogue of the equisingular displacements of complex spaces with "ordinary singularities".

§ 1. Logarithmic deformations of holomorphic maps. Let X be a compact complex manifold, C an analytic subset of X of simple normal crossing, and f a holomorphic map of X into a complex manifold Y.

Definition 1. By a family of logarithmic deformations of (X, C, f), we mean a 6-tuple $(\mathcal{X}, \mathcal{C}, \Phi, \pi, o, T)$ satisfying the following:

(1) $(\mathcal{X}_o, \mathcal{C}_o, \Phi_o)=(X, C, f)$ and $o \in T$,

(2) $(\mathcal{X}-\mathcal{C}, \mathcal{X}, \mathcal{C}, \pi, o, T)$ is a family of logarithmic deformations of $(X-C, X, C)$ (cf. [3]),

(3) $(\mathcal{X}, \Phi, \pi, T)$ is a family of holomorphic maps into Y (cf. [5]).

We define the concepts of equivalence and completeness of families of logarithmic deformations of holomorphic maps into Y as in [3] and [5].

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Let $(\mathcal{X}, \mathcal{C}, \Phi, \pi, o, T)$ be a family of logarithmic deformations of (X, C, f). Then we have a cohomological complex of sheaves on X of length 1: $\mathcal{L}_C: 0 \to \Theta_X(\log C) \xrightarrow{F} f^*\Theta_Y \to 0$, where $\Theta_X(\log C)$, Θ_Y and F denote the logarithmic tangent sheaf of (X, C) (cf. [3]), the tangent sheaf of Y, and the canonical homomorphism df respectively. Using the logarithmic coordinate on $\mathcal{X}$ (cf. [3]), we define the characteristic map

$$\tau: T_oT \longrightarrow \boldsymbol{H}^1(X; \mathcal{L}_C)$$

as in [5], where $\boldsymbol{H}^1(X; \mathcal{L}_C)$ is the first hypercohomology group of $\mathcal{L}_C$.

Theorem 1. *For any (X, C, f), there exists a family $(\mathcal{X}, \mathcal{C}, \Phi, \pi, o, T)$ of logarithmic deformations of holomorphic maps into Y such that*

(1) *$\tau: T_oT \to \boldsymbol{H}^1(X; \mathcal{L}_C)$ is injective,*

(2) *the family is complete at any point $t \in T$.*

Moreover the parameter space T is defined as follows: There exists a holomorphic map h of a neighbourhood of the origin of $\boldsymbol{H}^1(X; \mathcal{L}_C)$ into $\boldsymbol{H}^2(X; \mathcal{L}_C)$ such that $T = h^{-1}(0)$.

To prove this theorem in the case that C is not a divisor, we need the following propositions. Let $\tilde{X}$ be a monoidal transform of X with a canonical center of C (cf. [3]), D the total transform of C and $\tilde{f}$ the lifting of f.

Proposition 1.1. *For any family $(\tilde{\mathcal{X}}, \mathcal{D}, \tilde{\Phi}, \tilde{\pi}, o, S)$ of logarithmic deformations of $(\tilde{X}, D, \tilde{f})$, there exist an open neighbourhood S' of o and a family $(\mathcal{X}, \mathcal{C}, \Phi, \pi, o, S')$ of logarithmic deformations of (X, C, f) such that $\tilde{\mathcal{X}}_{|S'}$ is a monoidal transform of $\mathcal{X}$, $\mathcal{D}_{|S'}$ the total transform of $\mathcal{C}$ and $\tilde{\Phi}_{|S'}$ the lifting of Φ.*

This is a consequence of [1] or [2] Theorem 9.1.

Proposition 1.2. (1) *$\boldsymbol{H}^n(\tilde{X}; \mathcal{L}_D) \cong \boldsymbol{H}^n(X; \mathcal{L}_C)$ for $n \geq 0$.*

(2) *The isomorphism of* (1) *for $n=1$ commutes with the characteristic maps.*

Proof. Let $\eta: \tilde{X} \to X$ be the natural morphism. At first we have the following spectral sequence by considering the composite of the functors $\Gamma(X, *)$ and $\eta_*: E_2^{p,q} = H^p(X, \boldsymbol{R}^q\eta_*\mathcal{L}_D) \Longrightarrow \boldsymbol{H}^n(\tilde{X}; \mathcal{L}_D)$ where $\boldsymbol{R}^q\eta_*$ is the q-th hyperderived functor of η_*. Next we have $\boldsymbol{R}^q\eta_*\mathcal{L}_D = \boldsymbol{H}^q(\mathcal{L}_C)$ by [3] Theorem 2. Finally, through the spectral sequence $E_2^{p,q} = H^p(X, \boldsymbol{H}^q(\mathcal{L}_C)) \Longrightarrow \boldsymbol{H}^n(X; \mathcal{L}_C)$, we have the isomorphism (1). (2) follows by a direct calculation.

Proof of Theorem 1. By these propositions, we may assume that C is a divisor. Since $\Theta_X(\log C)$ is locally free if C is a divisor, we can construct the desired family by the method in [5]. Q.E.D.

The following theorem is proved analogously as [8] since $\boldsymbol{H}^0(X; \mathcal{L}_C) = 0$.

Theorem 2. *If f is a non-degenerate map, the family in Theorem 1 is the universal family.*

§2. Equisingular displacements of surfaces with ordinary singularities. Let W be a compact non-singular threefold and S be a hypersurface of W with only ordinary singularities (cf. [4]). We denote by P and Δ the set of all triple points of S and the double curve respectively. Let $\sigma_1: W_1 \to W$ be the monoidal transformation of W with the center P, Δ_1 the proper transform of Δ, and $\sigma_2: \hat{W}=W_2 \to W_1$ the monoidal transformation of W_1 along Δ_1, then the total transform D of S by the composition $f=\sigma_1 \circ \sigma_2$ is of simple normal crossing.

If $E=(\mathcal{S}, \pi, o, T)$ be a family of equisingular displacements of S in W (cf. [4]), then we have a family of logarithmic deformations $L(E)=(\hat{\mathcal{W}}, \mathcal{D}, \Phi, \hat{\pi}, o, T)$ of $(\hat{W}, D, f)$ by a succession of monoidal transformations of the above type.

Proposition 2.1. (1) *For any family of logarithmic deformations $L=(\hat{\mathcal{W}}, \mathcal{D}, \Phi, \hat{\pi}, o, T)$ of $(\hat{W}, D, f)$, there exists a family of equisingular displacements $E=(\mathcal{S}, \pi, o, T')$ of S in W parametrized by a neighbourhood T' of o in T such that $L(E)=L_{|T'}$,*

(2) *let $L(E_i)=L_i$ $(i=1, 2)$, then E_1 is equivalent to E_2 if and only if L_1 is equivalent to L_2,*

(3) *E is maximal at $t \in T$ if and only if L is complete at $t \in T$.*

Proof. (1) From Fujiki-Nakano [1] or Horikawa [2] Theorem 9.1, we infer that there exists a family $(\mathcal{W}, \tilde{\pi}, o, T')$ of deformations of W parametrized by a neighbourhood T' of o such that $\hat{\mathcal{W}}$ is obtained by a succession $\Psi=\tau_1 \circ \tau_2: \hat{\mathcal{W}} \to \mathcal{W}_1 \to \mathcal{W}$ of monoidal transformations with $\tau_{i,o}=\sigma_i$ $(i=1, 2)$. Since $\chi=\Phi_o \Psi^{-1}$ is a bimeromorphic map of $\mathcal{W}$ to $W \times T'$ over T' and $\chi_o=id_W$, χ induces a biholomorphic map of $\mathcal{W}_{|T''}$ onto $W \times T''$ over a neighbourhood T'' of o. Then we may assume that $\Phi: \hat{\mathcal{W}} \to W \times T$ is a succession of monoidal transformations of the same type as above. It is proved in [6] Ch. 3 that $(\mathcal{S}, \pi, o, T)$ is a family of equisingular displacements of S in W if we set $\mathcal{S}=\Phi(\mathcal{D})$, $\pi=p_{2|\mathcal{S}}$ and replace T by a smaller neighbourhood of o, where p_2 denotes the projection of $W \times T$ onto the second factor.

(2) and (3) are easily checked. Q.E.D.

Under the above situation, the following proposition is a consequeuce of [7].

Proposition 2.2. (1) $H^n(\hat{W}; \mathcal{L}_D) \cong H^{n-1}(S, \Phi_S)$ $(n \geqq 1)$.

(2) *The following diagram commutes:*

$$\begin{array}{ccc} & \overset{\tau}{\nearrow} & H^1(\hat{W}; \mathcal{L}_D) \\ T_oT & & \Vert \wr \\ & \underset{\sigma}{\searrow} & H^0(S, \Phi_S) \end{array}$$

where the characteristic map σ of $(\mathcal{S}, \pi, o, T)$ is referred to [4].

Hence we derive the following existence theorem of equisingular displacements of surfaces with ordinary singularities from Theorems 1 and 2.

Theorem 3. *For any* (W, S), *there exists a family* $(\mathcal{S}, \pi, o, T)$ *of equisingular displacements of* S *in* W *such that*

(1) $\mathcal{S}_o = S$,

(2) $\sigma_t : T_t T \to H^0(S_t, \Phi_{S_t})$ *is injective for any point* $t \in T$,

(3) *the family is universal at any point* $t \in T$.

Moreover the parameter space T *is defined as follows: There exists a holomorphic map* h *of a neighbourhood of the origin of* $H^0(S, \Phi_S)$ *into* $H^1(S, \Phi_S)$ *such that* $T = h^{-1}(0)$.

Proof. Theorems 1 and 2 assert the existence of the family $(\mathcal{S}, \pi, o, T)$ of equisingular displacements of S in W which is effective and universal at the reference point $o \in T$ and maximal at any point $t \in T$. Since $\mathcal{O}_{W \times T}([\mathcal{S}])$ is an invertible sheaf over $W \times T$ and Φ_{S_t} is a subsheaf of $\mathcal{O}_{W \times T}([\mathcal{S}])_{|S_t}$, the effectivity at any point $t \in T$ sufficiently close to o follows from the one at o. From these we also infer that the family is universal at $t \in T$ sufficiently close to o. Q.E.D.

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