6. *Meromorphic Solutions of Some Difference Equations of Higher Order*

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(Communicated by Kôsaku YOSIDA, M. J. A., Jan. 12, 1982)

1. Introduction. In this note, we will investigate the equation

(1.1) $$\alpha_n y(x+n)+\alpha_{n-1}y(x+n-1)+\cdots+\alpha_1 y(x+1)=R(y(x)),$$

where

(1.2) $$\begin{cases} R(w)=P(w)/Q(w), \\ P(w)=a_p w^p+\cdots+a_0, \\ Q(w)=b_q w^q+\cdots+b_0, \end{cases}$$

in which $\alpha_n, \cdots, \alpha_1$; $a_p, \cdots, a_0$; $b_q, \cdots, b_0$ are constants, $\alpha_n a_p b_q \neq 0$, $P(w)$ and $Q(w)$ are mutually prime. In the below, p and q denote the degrees of the nominator $P(w)$ and the denominator $Q(w)$ in (1.2), respectively. Put

(1.3) $$q_0=\max(p, q).$$

When $n=1$ in (1.1), we have

(1.1′) $$y(x+1)=R(y(x)).$$

If $q_0=1$ in (1.1′), then the equation reduces to a linear difference equation, by some linear transformation if necessary. When $q_0 \geqq 2$, equation (1.1′) is studied by Shimomura [3] and by the author [4]. Results are:

Proposition 1. *Suppose* $q_0 \geqq 2$. *Any nontrivial meromorphic solution of* (1.1′) *is transcendental and of infinite order* (in Nevanlinna's sense).

Proposition 2. *When* $q=0$ *and* $q_0=p\geqq 2$ *in* (1.1′), *any meromorphic solution is entire.*

Proposition 3. (1.1′) *possesses nontrivial meromorphic solutions.*

Now we consider the case $n>1$ in (1.1). It will be observed that several differences appear between the cases $n=1$ and $n>1$.

2. Transcendency and order. Prop. 1 does not hold for $n>1$. e.g.,

(2.1) $$y(x+2)-y(x+1)=-y(x)^2/[(1+2y(x))(1+y(x))]$$

has a rational solution $y(x)=1/x$. However, we have

Theorem 2.1. *When* $p>q\geqq 0$ *and* $q_0=p\geqq 2$, *then any meromorphic solution of* (1.1) *is transcendental.*

Proof. Suppose there would exist a rational solution $y(x)$ for (1.1).

When $q\geqq 1$. Let μ be a number such that $Q(\mu)=0$, and x_0 be such that $y(x_0)=\mu$. Obviously, $x_0\neq\infty$. Thus there is some k, $1\leqq k\leqq n$, such that x_0+k is a pole for $y(x)$. Put

$$k_1=\max\{k; 1\leqq k\leqq n,\ x_0+k \text{ is a pole for } y(x)\},$$
$$x_1=x_0+k_1.$$

Similarly, since $p>q$, there is k_2, $1\leqq k_2\leqq n$, such that x_1+k_2 is a pole for $y(x)$. Repeating this procedure, $y(x)$ would have an infinite number of poles, which contradicts the supposition of rationality.

When $q=0$. If $y(x)$ has a pole, then the above arguments apply, and we have a contradiction also. If $y(x)$ has no poles hence a polynomial, then, inserting it into (1.1) and comparing the degrees of polynomials on both sides, we also obtain a contradiction since $p\geqq 2$.

Q.E.D.

Let us give another counter-example to Prop. 1. The equation

$$y(x+2)+y(x+1)=[y(x)^2+1]/y(x) \tag{2.2}$$

has a transcendental meromorphic solution $y(x)=(e^{\pi ix}+1)/(e^{\pi ix}-1)$, which is of order 1. However, we have

Theorem 2.2. *Suppose $q_0>n$. Then any meromorphic solution of* (1.1) *is transcendental and of infinite order.*

Proof. We will show here the transcendency only. The fact that the order is ∞ has been proved by Ochiai [2].

In view of Theorem 2.1, we can suppose $p\leqq q$, hence $q_0=q$. Assume there would be a rational solution $y(x)=A(x)/B(x)$, in which $\deg[A(x)]=a$, $\deg[B(x)]=b$. We can suppose $b_0\neq 0$ in (1.2), by considering $y(x)+\beta$ $(Q(\beta)\neq 0)$ instead of $y(x)$, if necessary. Put

$$\alpha_nA(x+n)/B(x+n)+\cdots+\alpha_1A(x+1)/B(x+1)=C(x)/D(x),$$

where $\deg[D(x)]\leqq nb$, $\deg[C(x)]\leqq a+(n-1)b$. On the other hand

$$R(y(x))=B(x)^{q-p}[E(x)/F(x)],$$

where

$$E(x)=a_pA(x)^p+a_{p-1}A(x)^{p-1}B(x)+\cdots+a_0B(x)^p,$$
$$F(x)=b_qA(x)^q+b_{q-1}A(x)^{q-1}B(x)+\cdots+b_0B(x)^q.$$

$E(x)$ and $F(x)$ are obviouly mutually prime.

(i) Suppose $a<b$. Then $\deg[F(x)]=bq=bq_0>bn\geqq\deg[D(x)]$, which is a contradiction.

(ii) Suppose $a>b$. Then $\deg[E(x)]=ap+b(q-p)=(a-b)p+bq>a+b(n-1)\geqq\deg[C(x)]$, which is also a contradiction.

(iii) Suppose $a=b$. Then $\lim_{x\to\infty}[A(x)/B(x)]=\lambda\neq 0,\ \infty$. λ satisfies $(\alpha_n+\cdots+\alpha_1)\lambda=R(\lambda)$, whence $Q(\lambda)\neq 0$. Put $y(x)=u(x)+\lambda$. Then $u(x)=A_1(x)/B_1(x)$ satisfies the equation

$$\alpha_nu(x+n)+\cdots+\alpha_1u(x+1)=P_1(u(x))/Q_1(u(x)),$$

where $Q_1(0)=Q(\lambda)\neq 0$. Since $\deg[B_1(x)]=\deg[B(x)]>\deg[A_1(x)]$, we have a contradiction in this case also, by the case (i).

Thus we conclude that $y(x)$ can not be rational. Q.E.D.

3. The case $p-q\geqq 2$. We have

Theorem 3.1. *Any solution of* (1.1) *is entire if* $q=0$ *and* $p\geqq 2$.

Proof. Let $y(x)$ be a meromorphic solution of (1.1), and let $s(x_0)$ be the order of a pole x_0 for $y(x)$. $s(x_0)$ is a nonnegative integer.

Suppose $s(x_0)>0$ for some x_0. Then by (1.1) we know that

$$s_0=\max\{s(x_0+k);\ k=1, \cdots, n\}>0.$$

Obviously $s(x_0)\leqq s_0/p$, and

$$s(x_0-1)\leqq\max(s_0, s(x_0))/p=s_0/p.$$

Similarly $s(x_0-2)\leqq\max(s_0, s(x_0), s(x_0-1))/p=s_0/p$. In general

$$s(x_0-k)\leqq s_0/p, \qquad 0\leqq k\leqq n.$$

Put

$$s_1=\max\{s(x_0-k);\ k=1, \cdots, n\}\leqq s_0/p,$$
$$k_1=\max\{k;\ s(x_0-k)>0,\ 0\leqq k\leqq n\},$$
$$x_1=x_0-k_1.$$

Obviously, $k_1>0$. As in the above, we can easily see that

$$s(x_1-k)\leqq s_1/p\leqq s_0/p^2, \qquad 1\leqq k\leqq n.$$

Thus we obtain a sequence of integers $\{k_1, k_2, \cdots\}$, $k_j>0$, such that

$$x_j=x_{j-1}-k_j \qquad \text{satisfies } 0<s(x_j)<s_0/p^j,$$

which leads obviously to a contradiction. Thus $s(x_0)=0$ for any x_0, which means that $y(x)$ is entire. Q.E.D.

Remark. When $Q(w)$ in (1.2) has only one zero point, then (1.1) may possess an entire solution. For example,

$$y(x+2)+y(x+1)=[y(x)^6+1]/y(x)^2 \tag{3.1}$$

has solution $y(x)=\exp[(-2)^x]$. However, it is easy to see that, if $Q(w)$ has at least two distinct zero points, then any meromorphic solution of (1.1) can not be entire.

Theorem 3.2. *When* $p-q\geqq 2$ *in* (1.2), *then any meromorphic solution of* (1.1) *is of order* ∞. (For the case $p-q=1$, see the example (2.2).)

Proof. Let $y(x)$ be a meromorphic solution of (1.1). $y(x)$ is transcendental by Theorem 2.1. Write $t=p-q\geqq 2$.

(i) When $y(x)$ is entire. By the remark above, $Q(w)$ must be of the form $(w-b)^q (q\geqq 0)$, where b is a const. Then

$$R(w)=c_t w^t+\cdots+c_0+c_{-1}(w-b)^{-1}+\cdots+c_{-q}(w-b)^{-q}.$$

(When $q=0$, we set $c_{-j}=0$, $j\geqq 1$.) Let r be so large that $M(r)>2|b|$, where

$$M(r)=\max_{|x|=r}|y(x)|. \tag{3.2}$$

Let x_0 be a point such that $|x_0|=r$ and $|y(x_0)|=M(r)$. Then

$$\begin{aligned}|R(y(x_0))|&\geqq|c_t|M(r)^t-\cdots-|c_0|-|c_{-1}|(2/M(r))-\cdots-|c_{-q}|(2/M(r))^q\\&\geqq(1/2)|c_t|\,M(r)^t\end{aligned} \tag{3.3}$$

if r is sufficiently large. Since $\max_{|x|=r}|y(x+k)|\leqq M(r+k)\leqq M(r+n)$,

(3.4) $|\alpha_n y(x+n)+\cdots+\alpha_1 y(x+1)| \leqq (|\alpha_n|+\cdots+|\alpha_1|)M(r+n)$,
on $|x|=r$. By (3.3) and (3.4), we have $M(r+n) \geqq AM(r)^t$ for a const. A, i.e., $\log M(r+n) \geqq t \log M(r)+O(1)$. Therefore,

$$\log M(r+nk) \geqq t^k B \log M(r) \qquad \text{for a const. } B>0.$$

If we write $\rho=r+nk$, then

$$\log M(\rho) \geqq (t^{1/n})^\rho B[\log M(r)/t^r] \qquad \text{for } r_0 \leqq r \leqq r_0+n$$

with a sufficiently large r_0, which shows that the order of $y(x)$ is ∞.

(ii) When $y(x)$ has a pole x_0. Let $s(x_0)$ be the order of the pole x_0. Write $|x_0|=r$. By (1.1), there is a k, $1 \leqq k \leqq n$, such that $x_1=x_0+k$ is a pole of order $s(x_1) \geqq ts(x_0)$. In general, for any m, there are poles $x_1, \cdots, x_m$ such that $|x_1|<|x_2|<\cdots<|x_m|$, $|x_j| \leqq r+nj$ $(1 \leqq j \leqq m)$, of order $s(x_j) \geqq t^j s(x_0)$. Let $N(r, y(x))$ be the counting function of $y(x)$ (see [1, p. 165]). Then $N(r+nm, y(x)) \geqq A \times t^m$ with a const. A. Hence writing $r+nm=\rho$, we obtain

$$T(\rho, y(x)) \geqq N(\rho, y(x)) \geqq A_1 t^{\rho/n} \qquad \text{with a const. } A_1,$$

which shows that the order of $y(x)$ is ∞. Q.E.D.

In a subsequent paper, we will show that, if p and q are non-negative integers, $p \geqq q+1$, $\max(p, q) \leqq n$, then there is an equation of the form (1.1) which possessess a meromorphic solution of finite order.

References

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