57. Conformally Related Product Riemannian Manifolds

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Introduction. Let M and M^* be product Riemannian manifolds of dimension $n\geq 3$, and denote the structures by (M, g, F) and (M^*, g^*, G) respectively. The product structures F of M and G of M^* are different from the identity tensor I and satisfy the relations

$$F^2=I, \qquad G^2=I,$$

$$g(FX, FX)=g(X, X), \qquad g^*(GX^*, GX^*)=g^*(X^*, X^*)$$

for any vector X of M and any vector X^* of M^*. The integrability conditions of F and G in M and M^* are

$$\nabla_X F=0, \qquad \nabla^*_{X^*}G=0,$$

where we have denoted by ∇ and ∇^* covariant differentiations with respect to g and g^* respectively. A conformal diffeomorphism f of M to M^* is characterized by the change

$$g^*=\rho^{-2}g \tag{0.1}$$

of the metric tensors, where ρ is a positive valued scalar field.

Under a diffeomorphism f of M to M^*, the image of a quantity on M^* by the induced map f^* of f will be denoted by the same character as the original. The structures F and G are said to be *commutative* with one another at a point P of M under f if $FG=GF$ at P. In a previous paper [2], one of the present authors has proved the following

Theorem A. *If both product Riemannian manifolds M and M^* are complete, then there is no global non-homothetic conformal diffeomorphism of M to M^* such that the product structures F and G are not commutative under it at a point of M.*

As the contraposition of Theorem A, we can state

Theorem B. *Let both product Riemannian manifolds M and M^* be complete. If there exists a global non-homothetic conformal diffeomorphism f of M onto M^*, then the diffeomorphism f has to make the product structures F and G commutative everywhere in M.*

By virtue of Theorem B, we shall investigate product Riemannian manifolds admitting a global non-homothetic conformal diffeomorphism. The purpose of the present paper is to prove the following

Theorem. *Let both M and M^* be complete, connected and simply connected product Riemannian manifolds of dimension $n\geq 3$. If there is a global non-homothetic conformal diffeomorphism f of M onto M^*,* *then the underlying manifold of M and M^* is the product $N_1 \times N_0 \times N_2$ of three complete Riemannian manifolds N_1, N_0 and N_2 and the associated scalar field ρ with f depends on one part, say N_0, only. If the metric forms of N_1, N_0 and N_2 are denoted by ds_1^2, ds_0^2 and ds_2^2 respectively, then*

(1) *M is the product $M_1 \times N_2$, where M_1 is an irreducible complete Riemannian manifold, and the metric form of M is written as*

$$\rho^2 ds_1^2 + ds_0^2 + ds_2^2 \tag{0.2}$$

on the underlying manifold $N_1 \times N_0 \times N_2$, and

(2) *M^* is the product $N_1 \times M_2^*$, where M_2^* is an irreducible complete Riemannian manifold, and the metric form of M^* is written as*

$$ds_1^2 + \rho^{-2}(ds_0^2 + ds_2^2) \tag{0.3}$$

on the same underlying manifold $N_1 \times N_0 \times N_2$.

1. Preliminaries. Throughout the present paper, we assume that the differentiability of manifolds, diffeomorphisms and quantities is of class C^∞. Greek indices run on the range 1 to n. Let M be the product $M_1 \times M_2$ of two Riemannian manifolds M_1 and M_2 of dimension n_1 and n_2 respectively, $n_1 + n_2 = n$. The manifolds M_1 and M_2 are called *parts* of M. Let (x^h, x^p) be a separate coordinate system of M such that (x^h) and (x^p) are local coordinate systems of the parts M_1 and M_2 respectively. Here and hereafter Latin indices run on the ranges:

$$h, i, j, k, \cdots = 1, 2, \cdots, n_1;$$
$$p, q, r, s, \cdots = n_1 + 1, \cdots, n.$$

In such a coordinate system of M, the metric tensor $g = (g_{\mu\lambda})$ of M has pure components only, and the product structure $F = (F_\lambda^\kappa)$ has pure components $F_i^h = \delta_i^h$ and $F_q^p = -\delta_q^p$ only. The restrictions of the covariant differentiation ∇ of M on the parts M_1 and M_2 are expressed by ∇_i and ∇_q respectively. They are commutative with one another.

Under a conformal diffeomorphism f of M to M^*, we see that the induced tensor G from M^* to M constitutes an almost product Rimannian structure together with the metric g on M but is not necessarily integrable. The covariant tensor $G_{\mu\lambda}$ defined by $G_{\mu\lambda} = G_\mu^\kappa g_{\kappa\lambda}$ is symmetric in λ and μ. The product structures F and G are commutative under f if and only if G_λ^κ and $G_{\mu\lambda}$ have pure components only with respect to a separate coordinate system in M.

If the metric g^* of M^* is conformally related to g of M by (0.1), then the integrability condition $\nabla_\mu^* G_\lambda^\kappa = 0$ of G in M^* is equivalent to the differential equation

$$\nabla_\mu G_{\lambda\kappa} = -\rho^{-1}(G_{\mu\lambda}\rho_\kappa + G_{\mu\kappa}\rho_\lambda - g_{\mu\lambda}G_{\kappa\omega}\rho^\omega - g_{\mu\kappa}G_{\lambda\omega}\rho^\omega) \tag{1.1}$$

on M, where we have put $\rho_\lambda = \nabla_\lambda \rho$ and $\rho^\kappa = \rho_\kappa g^{\lambda\kappa}$.

We recall two lemmas in [1] of local versions.

Lemma 1. *A conformal diffeomorphism f of M onto M^* is* *homothetic if and only if* $\nabla_\mu G_{\lambda\kappa}=0$. *Then the structures* F *and* G *are commutative under* f. *In particular, if* f *preserves the product structures, that is,* $G=\pm F$, *then* f *is homothetic.*

Lemma 2. *If the structures* F *and* G *are commutative under a non-homothetic conformal diffeomorphism* f, *then the associated scalar field* ρ *is a function on either of the parts* M_1 *or* M_2 *only.*

We notice that, if the scalar field ρ is a function of both M_1 and M_2, then F and G are not commutative under f and hence there is no global non-homothetic conformal diffeomorphism having ρ as the associated scalar field.

Now we shall give Lemma 2 a global version as follows:

Lemma 3. *Let both* M *and* M^* *be product Riemannian manifolds. If the structures* F *and* G *are commutative under a non-homothetic conformal diffeomorphism* f *of* M *onto* M^*, *then the associated scalar field* ρ *is a function on either* M_1 *or* M_2 *only over the whole manifold* M.

Proof. By means of the commutativity of F and G, the structure G is pure and the equation (1.1) referred to a separate coordinate system splits into the following equations:

(1.2) $$\nabla_j G_{ih}=-\rho^{-1}(G_{ji}\rho_h+G_{jh}\rho_i-g_{ji}G_{hk}\rho^k-g_{jh}G_{ik}\rho^k);$$

(1.3) $$\nabla_q G_{ji}=0;$$

(1.4) $$\nabla_j G_{pi}=-\rho^{-1}(G_{ji}\rho_p-g_{ji}G_{pr}\rho^r)=0;$$

(1.5) $$\nabla_q G_{pi}=-\rho^{-1}(G_{qp}\rho_i-g_{qp}G_{ih}\rho^h)=0;$$

(1.6) $$\nabla_j G_{qp}=0;$$

(1.7) $$\nabla_r G_{qp}=-\rho^{-1}(G_{rq}\rho_p+G_{rp}\rho_q-g_{rq}\rho_{ps}\rho^s-g_{rp}G_{qs}\rho^s).$$

The equations (1.3) and (1.6) mean that $G_1=(G_i^h)$ depends on M_1 only and $G_2=(G_q^p)$ does on M_2 only.

If there are two points P and Q such that $\rho_i(P)\neq 0$ and $\rho_q(Q)\neq 0$, then it follows from (1.4) that G_1 is proportional to $I_1=(\delta_i^h)$ and hence $G_1=\pm I_1$ on the whole manifold M because of $G^2=I$ and the independence of G_1 on M_2. Similarly we have, from (1.5), $G_2=\pm I_2$ on the whole manifold M, where $I_2=(\delta_q^p)$. Since G is different from I, we have $G=\pm F$ on the whole manifold M and hence f is homothetic by Lemma 1. This contradicts to the non-homothety of f. Therefore the associated scalar field ρ should be a function dependent of one part only. Q.E.D.

2. Proof of Theorem. Since there exists a global non-homothetic conformal diffeomorphism f of M onto M^*, it follows from Theorem B that the product structures F and G are commutative everywhere in M and from Lemma 3 that ρ may be assumed as a function of M_1 only. As seen in the proof of Lemma 3, we have $G_2=\pm I_2$. Choose $G_2=I_2$. Then we have $G_{qp}=g_{qp}$ with respect to a separate coordinate system (x^h, x^p) in M.

We denote by $M_1(P)$ the part M_1 passing through any point P of M and by M_1' the image $f(M_1(P))$ of $M_1(P)$ by f. If we denote by $\overline{ds}_1^2$ and ds_2^2 the metric forms of M_1 and M_2 respectively, then the induced metric form of M_1' in M^* is identical with $\rho^{-2}\overline{ds}_1^2$. The part M_1 is simply connected, and so is the image M_1'. Since M^* is complete, the submanifold M_1' is also complete.

Since the equation (1.2) leads to the integrability condition $\nabla^* G_1 = 0$ of G_1 on M_1' and we have chosen $G_2 = I_2$, the structure G_1 on M_1' can be written in the form $G_1 = -I_1$ or

$$G_1 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}. \tag{2.1}$$

If $G_1 = -I_1$, then we have $G = \pm F$ and hence f is homothetic by Lemma 1. Thus G_1 must be of the form (2.1) on M_1'. It follows from (2.1) that M_1' is a product Riemannian manifold $N_1 \times N_0$ of two complete Riemannian manifolds N_1 and N_0. Since $G_{qp} = g_{qp}$, the equation (1.5) implies $G_i^h \rho_h = \rho_i$. From this equation and (2.1), we easily see that the associated scalar field ρ depends on N_0 only. If we denote by ds_1^2 and ds_0^2 the metric forms of N_1 and N_0 respectively, then the underlying manifold of the part M_1 of M is $N_1 \times N_0$ and the metric form $\overline{ds}_1^2$ is written as $\overline{ds}_1^2 = \rho^2(ds_1^2 + ds_0^2)$. Putting $N_2 = M_2$ and rewriting ds_0^2 in place of $\rho^2 ds_0^2$, we see that the underlying manifold of M is the product $N_1 \times N_0 \times N_2$ and the metric form is given by (0.2). The metric form of M^* is then expressed as (0.3) on the same underlying manifold $N_1 \times N_0 \times N_2$.

If M_1 is reducible and a Riemannian product $M_1^1 \times M_1^2$ of two Riemannian manifolds M_1^1 and M_1^2 and the associated scalar field ρ depends on both the parts M_1^1 and M_1^2, then we consider M as the product $M_1^1 \times (M_1^2 \times M_2)$ of the parts M_1^1 and $M_1^2 \times M_2$ in place of M_1 and M_2. Theorem A and the remark following Lemma 2 show that there is no global non-homothetic conformal diffeomorphism having such ρ as the associated scalar field. Hence M_1 is irreducible. Similarly we see that M_2^* with the underlying manifold $N_0 \times N_2$ is an irreducible part of M^*.

References

[1] Y. Tashiro: On conformal diffeomorphisms between complete product Riemannian manifolds. J. Math. Soc. Japan, **32**, 639–663 (1980).

[2] ——: On conformal diffeomorphisms between product Riemannian manifolds. Proc. Japan Acad., **57**A, 38–41 (1981).