53. Spectra of Domains with Small Spherical Neumann Boundary

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§1. **Introduction.** In this paper we consider the following problem: What can one say about the tones of a drum with a small tear?

Let Ω be a bounded domain in $\mathbf{R}^n$ with C^∞ boundary γ. And let B_ε be the ε-ball whose center is $w \in \Omega$. We put $\Omega_\varepsilon = \Omega \setminus \bar{B}_\varepsilon$. We consider the following eigenvalue problem:

(1.1)
$$-\Delta u(x) = \lambda(\varepsilon) u(x) \qquad x \in \Omega_\varepsilon$$
$$u(x) = 0 \qquad x \in \gamma$$
$$\frac{\partial u}{\partial \nu}(x) = 0 \qquad x \in \partial B_\varepsilon, \quad \partial/\partial\nu : \text{normal derivative.}$$

Let $0 < \lambda_1(\varepsilon) \le \lambda_2(\varepsilon) \le \cdots$ be the eigenvalues of $-\Delta$ counted by multiplicities. And let $0 < \lambda_1 \le \lambda_2 \le \cdots$ be the eigenvalues of $-\Delta$ in Ω under the Dirichlet condition on γ. We have the following

Theorem 1. *Assume $n=2$. Fix j. Assume that λ_j is simple, then*

(1.2)
$$\lambda_j(\varepsilon) = \lambda_j - 2\pi\varepsilon^2 |\operatorname{grad} \varphi_j(w)|^2 + \pi\lambda_j\varphi_j(w)^2\varepsilon^2 + O(\varepsilon^3 |\log \varepsilon|^2)$$

holds as ε tends to zero, where $\varphi_j(x)$ denotes the eigenfunction associated with λ_j satisfying

$$\int_\Omega \varphi_j(x)^2 dx = 1.$$

Remarks. The remainder term $O(\varepsilon^3 |\log \varepsilon|^2)$ in (1.2) is not uniform with respect to j. It should be remarked that $\lambda_j(\varepsilon) < \lambda_j$ (ε: small) when $\varphi_j(w) = 0$, $\operatorname{grad} \varphi_j(w) \ne 0$, which is false when we put the Dirichlet condition on ∂B_ε. See Courant-Hilbert [1]. Also see Uchiyama [8]. The problem (1.1) corresponds to the eigenvalue problem for a drum with a small tear ∂B_ε.

We make a historical remark. Rauch-Taylor [6] gave the convergence of the eigenvalues $\lambda_k(\varepsilon) \to \lambda_k$ for any k. Our Theorem 1 is improvement of their result in case $n=2$. For the case where we put the Dirichlet condition on ∂B_ε, many results concerning the asymptotic behaviour of the eigenvalues as $\varepsilon \to 0$ are obtained recently. See [2]–[5], etc.

In §2 we give a basic idea of the proof of Theorem 1. Details of this paper will be given elsewhere.

§2. **Ideas.** A key tool to prove an asymptotic formula is a function theoretic version of the Schiffer-Spencer formula. Let $G(x, y)$ be the Green function of the Laplacian in Ω under the Dirichlet condition on γ. And let $N_\varepsilon(x, y)$ be the Green function satisfying

$$\Delta N_\varepsilon(x, y) = -\delta(x-y) \qquad x, y \in \Omega_\varepsilon$$
$$N_\varepsilon(x, y)|_{x \in \gamma} = 0 \qquad y \in \Omega_\varepsilon$$
$$\frac{\partial}{\partial \nu_x} N_\varepsilon(x, y)|_{x \in \partial B_\varepsilon} = 0 \qquad y \in \Omega_\varepsilon,$$

where $\partial/\partial\nu_x$ denotes the exterior normal derivative on $\partial\Omega_\varepsilon \backslash \gamma$. We define two symbols:

$$\langle \nabla_w a(x, w), \nabla_w b(x, w) \rangle = \sum_{i=1}^{2} \frac{\partial}{\partial w_i} a(x, w) \frac{\partial}{\partial w_i} b(x, w)$$

for any $a, b \in C^1(\Omega \backslash \{w\})$. It should be remarked that $\langle \nabla_w, \nabla_w \rangle$ is invariant under an orthonormal transformation of the orthonormal coordinates (w_1, w_2).

$$\langle H_w a(x, w), H_w b(x, w) \rangle = \sum_{i,j=1}^{2} \frac{\partial^2}{\partial w_i \partial w_j} a(x, w) \frac{\partial^2}{\partial w_i \partial w_j} a(x, w).$$

It should be remarked that $\langle H_w a, H_w b \rangle|_{x, y \in \Omega \backslash \{w\}}$ is invariant under any orthonormal transformation of basis, if $a(x, w)$, $b(x, w) \in C^3(\Omega \backslash \{w\})$ and $\Delta_w a(x, w) = \Delta_w b(x, w) = 0$ for $x, y \in \Omega \backslash \{w\}$.

We put

$$p_\varepsilon(x, y) = G(x, y) + 2\pi\varepsilon^2 \langle \nabla_w G(x, w), \nabla_w G(y, w) \rangle$$
$$+ (\pi/2)\varepsilon^4 \langle H_w G(x, w), H_w G(y, w) \rangle.$$

Let G_ε (resp. P_ε) be the integral operator acting on $L^2(\Omega_\varepsilon)$ whose integral kernel function is $G_\varepsilon(x, y)$ (resp. $P_\varepsilon(x, y)$). We have the following two lemmas.

Lemma 1. *For a constant C independent of ε*
$$(2.1) \qquad \|G_\varepsilon - P_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^{2-s}$$
holds, where $s > 0$ is an arbitrary fixed constant.

Lemma 2. *Let χ_ε be the characteristic function of Ω_ε. Then*
$$(2.2) \qquad \|(G_\varepsilon - P_\varepsilon)(\chi_\varepsilon \varphi_j)\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^3 |\log \varepsilon|^2$$
holds for a constant C independent of ε.

Remarks. The estimate (2.1) is weak in the sense that the right hand side is not $o(\varepsilon^2)$, however we only use the weak type estimate and Lemma 2 in the proof of Theorem 1.

An essential difference between the proof of Theorem 1 and the proof of the asymptotic formula for eigenvalues considering under the Dirichlet condition on ∂B_ε lies in the estimate of the maximum norm of the solution of the following equations:
$$(2.3) \qquad \Delta u_\varepsilon(x) = 0 \qquad x \in \Omega_\varepsilon$$
$$(2.4) \qquad u_\varepsilon(x)|_\gamma = 0$$
$$(2.5) \qquad u_\varepsilon(x)|_{\partial B_\varepsilon} = t_\varepsilon(x), \qquad \max |t_\varepsilon(x)| \leq M(\varepsilon),$$

or

(2.6) $\dfrac{\partial}{\partial \nu} u_\varepsilon(x)|_{\partial B_\varepsilon} = S_\varepsilon(x)$, $\max |S_\varepsilon(x)| \leq N(\varepsilon)$.

For the case (2.3)–(2.5), we have the estimate

(2.7) $|u_\varepsilon(x)| \leq C |\log \varepsilon|^{-1} |\log |x-w|| M(\varepsilon)$.

See [2].   On the other hand, the estimate

(2.8) $|u_\varepsilon(x)| \leq C\varepsilon (1+|\log (|x-w|/\varepsilon)|) N(\varepsilon)$

for the solution of (2.3), (2.4) and (2.6) is obtained.   We can not apply the maximum principle to prove (2.8).   (2.8) is proved by the alternate convergence procedure (different but the similar idea as alternierendes Verfahren due to Schwarz [7]) for the two domains $R^3 \backslash \bar{B}_\varepsilon$ and Ω.

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