51. z-Transformation by the New Operator Methods

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§1. **Introduction.** In the theory of electrical engineering or telecommunication engineering, z-transformation technique is widely used for the analysis and synthesis of the sampled data system. The discrete time function which represents the sampled signal or the sampled data system is denoted by

$$f^*(t)=\sum_{n=0}^{\infty} f(nT)\delta(t-nT) \qquad (n=0,1,2,\cdots). \tag{1}$$

The z-transform $F(z)$ of the series $\{f(nT)\}$ is defined as the infinite sum of

$$F(z)=\sum_{n=0}^{\infty} f(nT)z^{-n} \qquad (n=0,1,2,\cdots), \tag{2}$$

where z is the complex variable such as $z=e^{sT}$ for the Laplace variable s. Hence we have

$$F(z)=\sum_{n=0}^{\infty} f(nT)e^{-sTn}=\mathcal{L}f^*(t), \tag{3}$$

and we know that the z-transform for $f(nT)$ is suited to the Laplace transform for $f^*(t)$. We denote (2) as $F(z)=z\{f(nT)\}$.

To obtain the formulae of z-transformation or the inverse z-transformation, several methods such as power series, partial fraction or residues theorems are explained in [4]. However, W. Jentsch [3], and S. Hayabara-S. Haruki [2] have obtained the tables of correspondence for elements of sequence space E, to elements of operator space Q. We call this theory "New Operator Method" (N.O.M.).

In this note, we will prove that the z-transformation is identified with N.O.M. in [2].

Therefore, we will have the following advantages in this direction.

(1) Understanding of the z-transformation is easily gained with the use of fundamental knowledge of the Laplace transformation and the function theory of complex variables.

(2) We can fill up the tables of N.O.M. by using the results of z-transformation which is useful to solve sequence equations or difference equations with variable coefficients.

§2. **z-transformation by the new operator methods.** We have obtained in [2] the operator expression $a_n(p)$ for the sequence $\{a_n\}$ as follows:

$$\{a_n\}=a_n(p)=\sum_{n=0}^{\infty} a_n p^{-n}. \tag{4}$$

By the definition of z-transformation

$$F(z)=z\{f(nT)\}=\sum_{n=0}^{\infty} f(nT)z^{-n}. \tag{2}$$

In comparison (2) with (4), we see that the results of z-transformation are obtained by putting $p=z$ into the formulae for $f_n(p)$ with additive parameter T. These relations are shown in Table of the appendix.

§ 3. **Algebraic derivative and algebraic integral.** T. Fényes-P. Kosik [1] have studied the operational calculus on the sequences by the basis of formula

$$\{a_n\}=\sum_{n=0}^{\infty} a_n(1+q)^{-n}, \quad q=(l-1)^{-1}, \quad l=\{1,1,\cdots,1,\cdots\}. \tag{5}$$

However S. Hayabara-S. Haruki [2] have studied it on the basis of the formula

$$\{a_n\}=\sum_{n=0}^{\infty} a_n u^n, \qquad u=\{0,1,0,\cdots,0,\cdots\}. \tag{6}$$

We shall prove that the formula (5) is equivalent to (6). By the equality

$$l(1-u)=\{1,1,\cdots,1,\cdots\}\cdot\{1,-1,0,\cdots,0,\cdots\}$$
$$=\{1,0,\cdots,0,\cdots\}=1,$$

$u=1-l^{-1}$, and $1+q=l/(l-1)=(1-l^{-1})^{-1}$, i.e., $u=(1+q)^{-1}$.

T. Fényes-P. Kosik [1] have obtained the following

Theorem 1. *For the operation D which is defined by*

$$D\{a_n\}=\{-na_n\}, \tag{7}$$

$$D[\{a_n\}+\{b_n\}]=D\{a_n\}+D\{b_n\}. \tag{8}$$

$$D[\{a_n\}\cdot\{b_n\}]=\{a_n\}\cdot D\{b_n\}+\{b_n\}\cdot D\{a_n\}. \tag{9}$$

Definition 1. This operation D is called "Algebraic Derivative".

Definition 2. The operation D for the operator a/b in Q, is defined by the formula

$$D(a/b)=[b\cdot Da-a\cdot Db]/(b\cdot b), \qquad a,b\in E. \tag{10}$$

Theorem 2. (1) *For the operators in Q, we have the same formulae as* (8) *and* (9).

(2) $D[\alpha]=0, \quad Dp=p, \quad Dp^r=rp^r \quad (r\text{: constant}).$ (11)

(3) $Df(p)=[(d/dp)f(p)]\cdot p.$ (12)

We shall introduce the algebraic integral for the inverse calculation of the operation D.

Definition 3. If there exists the operator $y\in Q$ such as $Dy=x$ for the given operator $x\in Q$, we call that y is the "Algebraic Integral" of x, and express it as

$$y=\int x. \tag{13}$$

We shall introduce a "Logarithmic Operator" $\log p$ such as

$$\int[\alpha]=\alpha\log p=\log p^{\alpha}, \tag{14}$$

whereas T. Fényes-P. Kosik did not introduce.

Then we have the following formulae and we use them successfully to solve sequence equations or difference equations.

$$\int p(p-\alpha)^{-1} = \log(p-\alpha) + [\beta], \tag{15}$$

$$\int p(p-\alpha)^{-(k+1)} = -k^{-1}(p-\alpha)^{-k} + [\beta] \qquad (k \neq 0). \tag{16}$$

§4. Application to the solution of difference equations. To solve the difference equations with constant coefficients, we can apply N.O.M. But for the difference equation with variable coefficients, it is necessary to express all terms of equations as operators in Q, hence we must fill up the table of the appendix, by using inversely the table of z-transformations.

If $z\{f(nT)\} = F(z)$, we have

$$z\{a^{nT} f(nT)\} = F(a^{-T} z), \tag{17}$$

$$z\{e^{anT} f(nT)\} = F(e^{-aT} z), \tag{18}$$

hence

$$f_n \quad (\text{in } E) = f_n(p) \quad (\text{in } Q)$$

$$a^n f_n = f_n(a^{-1} p), \tag{19}$$

$$e^{an} f_n = f_n(e^{-a} p). \tag{20}$$

Thus we have the following table of the appendix. Also inversely, we have for the z-transformation

$$z\{(nT)^k f(nT)\} = (-Tz(d/dz))^k F(z), \tag{21}$$

by the formula

$$n^k f_n = (-D)^k f_n(p), \qquad Df_n(p) = [(d/dp) f_n(p)]p. \tag{22}$$

Appendix: Table for N.O.M. and z-Transformation

	E-space	Q-space	$f(nT)$	$F(z)=z\{f(nT)\}$	Domain of conv.
1	$[1]$ constant	1	$\delta_p(nT)$	1	$\|z\|<\infty$
2	$[\alpha]$	α	$\alpha\delta_p(nT)$	α	$\|z\|<\infty$
3	$\{1\}$	$l, p/(p-1)$	$\begin{cases} 1 & (n \geqq 0) \\ 0 & (n<0) \end{cases}$	$z/(z-1)$	$\|z\|>1$
4	$\{n\}$	$l^2-l, p/(p-1)^2$	nT	$Tz/(z-1)^2$	$\|z\|>1$
5	$\{n^2\}$	$p(p+1)/(p-1)^3$	n^2T	$Tz(z+1)/(z-1)^3$	$\|z\|>1$
6	$\{\alpha^n\}$	$p/(p-\alpha)$	α^{nT}	$z/(z-\alpha^T)$	$\|z\|>\alpha^T$
7	$\{\binom{n}{k}\alpha^{n-k}\}$	$p/(p-\alpha)^{k+1}$	$\binom{n}{k}\alpha^{(n-k)T}$	$z/(z-\alpha^T)^{k+1}$	$\|z\|>\alpha^T$
8	$\{e^{an}\}$	$p/(p-e^a)$	e^{anT}	$z/(z-e^{aT})$	$\|z\|>e^{aT}$
9	$\{a_{n+1}\}$	$pa_n - p[a_0]$	$f\{(n+1)T\}$	$zF(z)-zf(0)$	
10	$\{a_{n+2}\}$	$p^2a_n - p^2[a_0] - p[a_1]$	$f\{(n+2)T\}$	$z^2F(z)-z^2f(0)$ $-zf(T)$	
11	$\{a_{n+k}\}$	$p^k a_n - p^k[a_0]$ $-p^{k-1}[a_1] - \cdots$ $-p[a_{k-1}]$	$f\{(n+k)T\}$	$z^kF(z) - z^kf(0)$ $-z^{k-1}f(T) - \cdots$ $-zf\{(k-1)T\}$	
12	$\{f_{mn}\}$	$f_n(p^{1/m})$	$f(mnT)$	$F(z^{1/m})$	

m, n: pos. integer

13	$\{a^n f_n\}$	$f_n(a^{-1}p)$	$a^{nT}f(nT)$	$F(a^{-T}z)$
14	$\{e^{an}f_n\}$	$f_n(e^{-a}p)$	$e^{anT}f(nT)$	$F(e^{-aT}z)$
15	$\{nf_n\}$	$-Df_n=-pf_n'(p)$	$(nT)f(nT)$	$-TzF'(z)$
16	$\{n^2 f_n\}$	$(-D)^2 f_n = p^2 f_n''(p) + pf_n'(p)$	$(nT)^2 f(nT)$	$T^2z^2F''(z) + TzF'(z)$
17	$\{n^k f_n\}$	$(-D)^k f_n = \left(-p\dfrac{d}{dp}\right)^k f_n(p)$	$(nT)^k f(nT)$	$\left(-Tz\dfrac{d}{dz}\right)^k F(z)$

Example 1. Find a general expression of x_n satisfying

$$(n+1)ax_n - nx_{n+1} = a^{n+1}, \tag{1}$$

$x_0=1$, (a is a constant).

Solution. By $Dx_n=-nx_n$, $x_{n+1}=px_n-p\cdot 1$, (1) is reduced to

$$a(-Dx_n)+ax_n+D(px_n-p)=ap/(p-a),$$
$$(p-a)Dx_n+(p+a)x_n=p+ap(p-a)^{-1},$$
$$Dx_n+(p+a)(p-a)^{-1}x_n=p(p-a)^{-1}+ap(p-a)^{-2}. \tag{2}$$

For

$$Dx_n+(p+a)(p-a)^{-1}x_n=0, \tag{3}$$
$$Dx/x=-(p+a)(p-a)^{-1}=1-2p(p-a)^{-1},$$
$$\log x=\log p-2\log(p-a)+c_0, \qquad \tilde{x}=cp(p-a)^{-2}. \tag{4}$$

For the particular solution x_p for (2),

$$x_p=\tilde{x}\int \tilde{x}^{-1}[p(p-a)^{-1}+ap(p-a)^{-2}]=p(p-a)^{-2}\int p=p^2(p-a)^{-2}$$
$$=p(p-a)^{-1}+ap(p-a)^{-2}. \tag{5}$$

Hence, by (4), (5) the general solution of (1) in E is

$$x=\{cna^{n-1}+(n+1)a^n\}. \tag{6}$$

By taking n term, we have

$$x_n=cna^{n-1}+(n+1)a^n \qquad (n \text{ is a constant}). \tag{7}$$

Particularly we have $x_n=a^n$, when $c=-a$.

References

[1] T. Fényes and P. Kosik: The algebraic derivative and integral in the discrete operational calculus. Studia Sci. Math. Hungar., 7, 117–130 (1972).

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[4] N. Kozima and T. Shinozaki: Introduction to z-Transformation. Tokai Univ. Press (1981) (in Japanese).