49. *Remarks on the Uniqueness in an Inverse Problem for the Heat Equation. II**)

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For $p \in C^1[0,1]$, $h \in \mathcal{R}$, $H \in \mathcal{R}$ and $a \in L^2(0,1)$, let $(E_{p,h,H,a})$ denote the heat equation

$$(1)\qquad \frac{\partial u}{\partial t}+\left(p(x)-\frac{\partial^2}{\partial x^2}\right)u=0 \qquad (0<t<\infty,\ 0<x<1),$$

with the boundary condition

$$(2)\qquad \frac{\partial u}{\partial x}-hu|_{x=0}=\frac{\partial u}{\partial x}+Hu|_{x=1}=0 \qquad (0<t<\infty),$$

and with the initial condition

$$(3)\qquad u|_{t=0}=a(x) \qquad (0<x<1).$$

Let $A_{p,h,H}$ be the realization in $L^2(0,1)$ of the differential operator $p(x)-\partial^2/\partial x^2$ with the boundary condition (2), and let $\{\lambda_n\}_{n=0}^\infty$ and $\{\phi_n\}_{n=0}^\infty$ be the eigenvalues and the eigenfunctions of $A_{p,h,H}$, respectively, the latter being normalized by $\|\phi_n\|_{L^2(0,1)}=1$. Noting that each λ_n is simple $(-\infty<\lambda_0<\lambda_1<\cdots\to\infty)$, we call $N\equiv\sharp\{\lambda_n\,|\,(a,\phi_n)=0\}$ the "degenerate number" of $a\in L^2(0,1)$ with respect to $A_{p,h,H}$, where $(\ ,\ )$ means the L^2-inner product.

Henceforth p, h, H and a are given, $u=u(t,x)$ is the solution of $(E_{p,h,H,a})$ and N is the degenerate number of a with respect to $A_{p,h,H}$. Take T_1, T_2 in $0\leqq T_1<T_2<\infty$ and set

$$\mathcal{M}^0=\mathcal{M}^0_{p,h,H,a,x_0}\equiv\{(q,j,J,b)\in C^1[0,1]\times\mathcal{R}\times\mathcal{R}\times L^2(0,1)\,|$$

$$(4')\qquad v(t,0)=u(t,0),\quad v(t,x_0)=u(t,x_0),\quad v_x(t,x_0)=u_x(t,x_0)$$

$$(T_1\leqq t\leqq T_2)$$

holds for the solution $v=v(t,x)$ of the equation $(E_{q,j,J,b})\}$.

Clearly $(p,h,H,a)\in\mathcal{M}^0$. In the previous work [7], the author showed

Theorem 0. (i) *In the case of* $x_0=1$,

$$(5')\qquad \mathcal{M}^0=\{(p,h,H,a)\}$$

if and only if $N=0$.

(ii) *In the case of* $1/2<x_0<1$, *(5′) holds whenever* $N<\infty$.

(iii) *In the case of* $x_0=1/2$, *(5′) holds if and only if* $N\leqq 1$.

(iv) *In the case of* $0<x_0<1/2$, *we always have* $\mathcal{M}^0\supsetneqq\{(p,h,H,a)\}$.

In the present paper, we consider

$$\mathcal{M}^1=\mathcal{M}^1_{p,h,H,a,x_0}\equiv\{(q,j,J,b)\in C^1[0,1]\times\mathcal{R}\times\mathcal{R}\times L^2(0,1)\,|$$

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*) This work was supported partly by the Fûju-kai.

$$(4)\qquad v(t, x_0)=u(t, x_0), \quad v_x(t, x_0)=u_x(t, x_0) \quad (T_1 \leqq t \leqq T_2)$$

holds for the solution $v=v(t, x)$ of the equation $(E_{q,j,J,b})\}$, and study when

$$(5)\qquad \mathcal{M}^1=\{(p, h, H, a)\}$$

is satisfied. We note that (5) holds only if $x_0=1/2$ and $N \leqq 1$ by Theorem 0.

Our results are the following

Theorem 1. *(5) holds true if* $x_0=1/2$ *and* $N=0$.

Theorem 2. *Let* $x_0 \neq 1/2$ *and assume* $1/2<x_0<1$ *without loss of generality. Then, (4) implies* $J=H$ *and* $q(x)=p(x)$ $(x_0 \leqq x \leqq 1)$, *whenever* $N<\infty$.

Theorem 3. *Under the same situation as in Theorem 2, we have*

$$(6)\qquad \mathcal{M}^1_{p,h,H,a,x_0} \cap \{(q, j, J, b) \in C^1[0,1] \times \mathcal{R} \times \mathcal{R} \times L^2(0,1) \mid q(x)=p(x) \quad (1/2 \leqq x \leqq x_0)\}=\{(p, h, H, a)\},$$

if and only if $N \leqq 1$.

In view of Theorems 1–3, we call $(x_0, 1)$ the "domain of uniqueness" in the case of $1/2<x_0<1$, which turns out to be $(0, x_0)$ in the case of $0<x_0<1/2$. The proof of Theorems 1–3 is based on

Lemma 1. *Put* $D_1=\{(x, y) \mid 1/2<x<1, 1-x<y<x\}$. *Then for each* $p \in C^1[0,1]$ *and* $q \in C^1[1/2, 1]$, *there exists a unique* $K \in C^2(\bar{D}_1)$ *such that*

$$(7.a)\qquad K_{xx}-K_{yy}+p(y)K=q(x)K \qquad ((x, y) \in \bar{D}_1)$$

$$(7.b)\qquad K(x, x)=1/2\int_{1/2}^{x}(q(s)-p(s))ds \qquad (1/2 \leqq x \leqq 1)$$

$$(7.c)\qquad K(x, 1-x)=0 \qquad (1/2 \leqq x \leqq 1).$$

Lemma 2. *Let* $\Phi=\Phi(x) \in C^2[0,1]$ *satisfy*

$$(8)\qquad (p(x)-d^2/dx^2)\Phi=\lambda\Phi \qquad (0 \leqq x \leqq 1)$$

for some $\lambda \in \mathcal{R}$. *Then*

$$(9)\qquad \Psi(x)=\Phi(x)+\int_{1-x}^{x}K(x, y)\Phi(y)dy \qquad (1/2 \leqq x \leqq 1)$$

satisfies

$$(10)\qquad \Psi(1/2)=\Phi(1/2), \quad \Psi'(1/2)=\Phi'(1/2), \quad (q(x)-d^2/dx^2)\Psi=\lambda\Psi \quad (1/2 \leqq x \leqq 1).$$

The proof will be given in a forthcoming paper along with a detailed proof of Theorem 0. For other references on the inverse problems for the heat equation, see Suzuki [5]. See also Seidman [3], Pierce [2], Suzuki-Murayama [8], Murayama [1] and Suzuki [4], [6].

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