45. *On Totally Geodesic Hermitian Symmetric Submanifolds of Kähler Manifolds*

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(Communicated by Kunihiko KODAIRA, M. J. A., April 12, 1982)

1. Introduction. Let M be a Kähler manifold with a Kähler metric g and a complex structure J. We denote by $Aut^0(M)$ the identity component of the group of all holomorphic isometries of M and by $\mathfrak{g}(M)$ the Lie algebra of $Aut^0(M)$. For each $X \in \mathfrak{g}(M)$, X^* means the vector field on M generated by $\{\exp tX\}_{t\in R}$. Then the correspondence: $X \to X^*$ can be extended to a linear mapping of $\mathfrak{g}(M)^c$, the complexification of $\mathfrak{g}(M)$, to the Lie algebra $\mathfrak{X}(M)$ of all vector fields on M by putting $(\sqrt{-1}X)^* = JX^*$ for $X \in \mathfrak{g}(M)$. We set for $p \in M$

$$\mathfrak{b}^1(p) = \{X \in \mathfrak{g}(M)^c\,;\ j_p^1(X^*) = 0\},$$

where $j_p^1(X^*)$ denotes the 1-jet of X^* at p. If M is a hermitian symmetric space, then $\dim M = \dim \mathfrak{b}^1(p)$ for any $p \in M$. In this paper, we shall prove the following

Theorem. *Let M be a Kähler manifold. For each point $p \in M$, there exists a totally geodesic hermitian symmetric submanifold $M(p)$ through p such that*

(*a*) $\dim M(p) = \dim \mathfrak{b}^1(p)$.

(*b*) *Let f be a holomorphic isometry of M and $q = f\cdot p$. Then $f\cdot M(p) = M(q)$.*

2. Let K_p be the isotropy subgroup of $Aut^0(M)$ at p and let $\mathfrak{k}_p$ be the Lie algebra of K_p. We set

$$\mathfrak{m}(p) = \{\text{the real part of } X\,;\ X \in \mathfrak{b}^1(p)\}.$$

Since $\mathfrak{b}^1(p)$ is an $\mathrm{Ad}\,K_p$-invariant complex subspace, $\mathfrak{m}(p)$ is an $\mathrm{Ad}\,K_p$-invariant subspace of $\mathfrak{g}(M)$ and $\mathfrak{m}(p) = \{\text{the imaginary part of } X\,;\ X \in \mathfrak{b}^1(p)\}$.

For each $\xi \in \mathfrak{X}(M)$, we denote by A_ξ the tensor field of type (1, 1) defined by

$$A_\xi v = -\nabla_v \xi, \qquad \text{for } v \in T_p(M),$$

where ∇ denotes the riemannian connection. Note that $A_\xi = \mathcal{L}_\xi - \nabla_\xi$.

Lemma 1. *For every $X \in \mathfrak{m}(p)$, $(A_{X^*})_p = 0$.*

Proof. There exists $Y \in \mathfrak{m}(p)$ such that $X + \sqrt{-1}Y \in \mathfrak{b}^1(p)$. Then for any $v \in T_p(M)$, $(A_{X^*+JY^*})_p v = -\nabla_v(X^* + JY^*) = 0$. Since X^* and Y^* are infinitesimal isometries, both $(A_{X^*})_p$ and $(A_{Y^*})_p$ are skew-symmetric with respect to g. Let $\xi \in \mathfrak{X}(M)$. Then $J \circ (A_{Y^*})_p \xi_p = J[Y^*, \xi]_p - J(\nabla_{Y^*}\xi)_p$

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$=(A_{Y^*})_p \circ J\xi_p$ because $\mathcal{L}_{Y^*}J=0$ and $\nabla J=0$. Then for any $u, v \in T_p(M)$, $g((A_{JY^*})_p u, v) = g(-\nabla_u(JY^*), v) = g(J \circ (A_{Y^*})_p u, v) = -g((A_{Y^*})_p u, Jv) = g(u, (A_{Y^*})_p \circ Jv) = g(u, J \circ (A_{Y^*})_p v) = g(u, (A_{JY^*})_p v)$. Therefore $(A_{JY^*})_p$ is symmetric with respect to g and hence $(A_{X^*})_p=(A_{JY^*})_p=0$. Q.E.D.

Lemma 2. (a) $\mathfrak{m}(p) \cap \mathfrak{k}_p=0$ *and* $[\mathfrak{m}(p), \mathfrak{m}(p)] \subset \mathfrak{k}_p$.

(b) *There exists a unique complex structure I of $\mathfrak{m}(p)$ satisfying $(IX)^*_p=JX^*_p$ and the correspondence: $X \to X+\sqrt{-1}IX$ gives an isomorphism between $\mathfrak{m}(p)$ and $\mathfrak{b}^1(p)$.*

Proof. Let $X \in \mathfrak{m}(p) \cap \mathfrak{k}_p$. Then $(A_{X^*})_p$ is the linear isotropy representation of X at p. From Lemma 1, we know $(A_{X^*})_p=0$ and hence $X=0$. Next we take $X, Y \in \mathfrak{m}(p)$. Then $[X, Y]^*_p=-[X^*, Y^*]_p=(\nabla_{Y^*}X^*)_p=-(\nabla_{X^*}Y^*)_p=-(A_{X^*})_p Y^*_p+(A_{Y^*})_p X^*_p=0$, proving (a).

We know form (a) that for each $X \in \mathfrak{m}(p)$ there exists a unique element Y of $\mathfrak{m}(p)$ such that $X+\sqrt{-1}\,Y \in \mathfrak{b}^1(p)$. If we define an endomorphism I of $\mathfrak{m}(p)$ by $IX=Y$, then $(X+\sqrt{-1}IX)^*_p=X^*_p+J(IX)^*_p=0$. Therefore we get (b). Q.E.D.

Lemma 3. *For each $X \in \mathfrak{m}(p)$, we set $\gamma(t)=\exp tX \cdot p$. Then $\gamma(t)$ is a geodesic.*

Proof. Since X is an infinitesimal isometry, $\nabla_{X^*}(A_{X^*})=R(X^*, X^*)=0$, where R denotes the curvature tensor (cf. P. 235, [3]). Therefore the tensor field A_{X^*} is parallel along $\gamma(t)$. We have $(A_{X^*})_{\gamma(t)}=0$ because $(A_{X^*})_p=0$. Hence $\nabla_{\dot\gamma(t)}\dot\gamma(t)=-(A_{X^*})_{\gamma(t)}X^*=0$. Q.E.D.

3. We can now prove Theorem. By (a) of Lemma 2, $\mathfrak{l}=\mathfrak{k}_p+\mathfrak{m}(p)$ is a subalgebra of $\mathfrak{g}(M)$. Let L denote the connected subgroup of $Aut^0(M)$ corresponding to $\mathfrak{l}$. We put

$$M(p)=L\cdot p=L/L \cap K_p.$$

Note that $L \cap K_p$ is compact because the Lie algebra of $L \cap K_p$ is equal to $\mathfrak{k}_p$. By (b) of Lemmas 2 and 3, $M(p)$ becomes a totally geodesic complex submanifold of M. Let N be the closed subgroup of L defined by $N=\{a \in L\,;\, a\cdot q=q$ for any $q \in M(p)\}$. N is a normal subgroup of L contained in $L \cap K_p$ and the Lie algebra $\mathfrak{n}$ of N is an ideal of $\mathfrak{l}$ satisfying $\mathfrak{n}=\{X \in \mathfrak{k}_p\,;\,[X, \mathfrak{m}(p)]=0\}$. We put $L'=L/N$, $K'=L \cap K_p/N$ and $\mathfrak{l}'=\mathfrak{l}/\mathfrak{n}=\mathfrak{k}_p/\mathfrak{n}+\mathfrak{m}(p)$. Then $M(p)=L'/K'$. The automorphism σ of $\mathfrak{l}$ defined by $\sigma|_{\mathfrak{k}_p}=1$ and $\sigma|_{\mathfrak{m}(p)}=-1$ induces an involutive automorphism σ' of $\mathfrak{l}'$ and the pair $(\mathfrak{l}', \sigma')$ is an effective orthogonal symmetric Lie algebra (cf. P. 229, [1]).

Let $\tilde{L}$ be the universal covering group of L' with the covering map $\omega: \tilde{L} \to L'$ and let $\tilde{K}=\omega^{-1}(K')$. We denote by $\tilde{K}^0$ the identity component of $\tilde{K}$. Then $\tilde{L}/\tilde{K}^0$ is a simply connected hermitian symmetric space and we can obtain the decompositions $\tilde{L}=L_0 \times L_- \times L_+$ and $\tilde{K}^0=K_0 \times K_- \times K_+$ in such a way that L_0/K_0, L_-/K_- and L_+/K_+ are hermitian symmetric spaces of the Euclidian type, compact type and non-compact type respectively. Let π_- and π_+ be the projections: $\tilde{L}\to L_-$ and $\tilde{L}\to L_+$ respectively. It is easy to see that the Lie algebras of $\pi_-(\tilde{K})$ and $\pi_+(\tilde{K})$ are those of K_- and K_+. Since $(L_+, \pi_+(\tilde{K}))$ is a pair associated with an orthogonal symmetric Lie algebra of non-compact type, $\pi_+(\tilde{K})$ is connected and hence $\pi_+(\tilde{K})=K_+$ (cf. P. 253, [1]). Note that $\pi_-(\tilde{K})$ is a compact subgroup of L_- (cf. P. 282, [1]). Then the homogeneous space $L_-/\pi_-(\tilde{K})$ has a Kähler metric such that the covering map: $L_-/K_-\to L_-/\pi_-(\tilde{K})$ is isometric. Since L_-/K_- is a hermitian symmetric space of compact type, the Ricci tensor of $L_-/\pi_-(\tilde{K})$ is positive definite and hence $L_-/\pi_-(\tilde{K})$ is simply connected (Kobayashi [2]). As a result, $\pi_-(\tilde{K})$ is conneced and hence $\pi_-(\tilde{K})=K_-$. We thereby obtain $\tilde{K}=\tilde{K}\cap L_0\times K_-\times K_+$ and $M(p)=L_0/\tilde{K}\cap L_0\times L_-/K_-\times L_+/K_+$. It remains to show that $L_0/\tilde{K}\cap L_0$ is symmetric. We write $L_0/\tilde{K}\cap L_0 =\Gamma\backslash \boldsymbol{C}^n$, where Γ is a discrete subgroup of holomorphic isometries of $\boldsymbol{C}^n(=L_0/K_0)$. Since L_0 contains all translations of $\boldsymbol{C}^n$, each element of Γ commutes with all translations. As a consequence Γ consists of translations and hence $L_0/\tilde{K}\cap L_0$ is symmetric. By construction, $\dim M(p)=\dim \mathfrak{b}^1(p)$. Let f be a holomorphic isometry of M and $q=f\cdot p$. Clearly $\mathrm{Ad} f K_p=K_q$ and $\mathrm{Ad} f\, \mathfrak{b}^1(p)=\mathfrak{b}^1(q)$. Therefore $\mathrm{Ad} f\, \mathfrak{m}(p)=\mathfrak{m}(q)$ and hence $M(q)=f\cdot M(p)$, completing the proof.

Remark. We can show that $M(p)$ is locally symmetric more directly from the following fact: Let ξ be an infinitesimal affine transformation of a manifold M with an affine connection ∇. If $(A_\xi)_p=0$. Then $\nabla_{\xi_p}R=(\mathcal{L}_\xi R)_p-(A_\xi R)_p=0$, where R denotes the curvature tensor. Similarly we get $\nabla_{\xi_p}T=0$ for the torsion tensor T.

As an immediate corollary of the proof of Theorem, we have

Theorem 4. *Let M be a connected Kähler manifold. Assume that there exists a point $p\in M$ such that $\dim \mathfrak{b}^1(p)=\dim M$. Then M is a hermitian symmetric space.*

Proof. Let $M(p)$ be the submanifold of M constructed in the proof of Theorem. Then $M(p)$ is open. Hence there exists $\varepsilon>0$ such that the ε-neighborhood U of p contained in $M(p)$. Let $q\in\overline{M(p)}$. There exists $p'\in M(p)$ such that $d(p',q)<\varepsilon$, where d denotes the distance function. Since $M(p)=L\cdot p$, there exists $f\in L$ such that $f\cdot p=p'$. Clearly $f^{-1}\cdot q\in U$. Therefore there exists $f'\in L$ such that $f^{-1}\cdot q=f'\cdot p$. Then $q=f\cdot f'\cdot p$ and hence $\overline{M(p)}=M(p)$. Q.E.D.

Remark. In the case where M is a Siegel domain of the second kind, our hermitian symmetric submanifold $M(p)$ is holomorphically isomorphic to the symmetric Siegel domain S constructed in [4].

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