43. *The Exponential Calculus of Microdifferential Operators of Infinite Order. II*

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1. Introduction. The purpose of this note is to give the exponential law for symbols of micro (=pseudo) differential operators or, more precisely, of holomorphic microlocal operators. We calculate $r(x,\xi)$ which satisfies

(1.1) $$:\exp\{p(x,\xi)\}:\ :\exp\{q(x,\xi)\}:=\ :\exp\{r(x,\xi)\}:.$$

Here the left-hand side is the composite operator of $:\exp\{p(x,\xi)\}:$ and $:\exp\{q(x,\xi)\}:$ whose symbols are $\exp\{p(x,\xi)\}$ and $\exp\{q(x,\xi)\}$ respectively (see [2] for the notation). Such $r(x,\xi)$ is expressed in a sum of symbols $\sum_{j=0}^{\infty} r_j(x,\xi)$. The first three terms were computed in our previous note [2] under suitable growth conditions. Now all $r_j(x,\xi)$ can be calculated from $p(x,\xi)$ and $q(x,\xi)$ without assuming any growth condition.

2. Formal symbols. In [2], we used the concept "formal symbol". But we did not give the precise definition of it there. Hence first we have to give it here.

Definition 1. Let Ω be a conic neighborhood of $\mathring{x}^*$ in T^*X. Here X is an open set in $\boldsymbol{C}^n$. Let

(2.1) $$P(t;x,\xi)=\sum_{j=0}^{\infty} t^j P_j(x,\xi)$$

be a formal power series in t with coefficients $P_j(x,\xi)$ $(j=0,1,2,\cdots)$ holomorphic in Ω. The formal series $P(t;x,\xi)$ is said to be a formal symbol defined in Ω if for any $\Omega'\subset\Omega$ there are positive constants R, A $(0<A<1)$ so that for each $h>0$ there exists $C>0$ such that

(2.2) $$|P_j(x,\xi)|\le CA^j\exp(h|\xi|)$$

for $(x,\xi)\in\Omega'$, $|\xi|\ge(j+1)R$, $j=0,1,2,\cdots$.

Remark. A formal symbol in the sense of [2] (cf. [1], [3]) is of course a formal symbol in the preceding meaning.

The addition and the multiplication for formal symbols are defined as those of formal power series in t. It is clear that the set of all formal symbols defined near $\mathring{x}^*$ forms a commutative ring E_{x^*}. There is an additive homomorphism

(2.3) $$E_{\mathring{x}^*}\ni P(t;x,\xi)\longmapsto\ :P(t;x,\xi):\ \in\mathcal{E}^R_{\mathring{x}^*}.$$

We often abbreviate $:\sum_{j=0}^{\infty} t^j P_j(x,\xi):$ to $:\sum_{j=0}^{\infty} P_j(x,\xi):$. The kernel of this homomorphism is not trivial. But we do not argue here about it (cf. [3]).

Proposition 2. *Let* $\sum_{j=0}^{\infty} t^j p_j(x, \xi)$ *be a formal symbol satisfying the following estimates. For any* $\Omega' \subset\subset \Omega$ *there are positive constants* R, A $(0<A<1)$ *so that for each* $h>0$ *there exists* $H>0$ *such that*

$$|p_j(x, \xi)| \leq A^j(h|\xi|+H) \tag{2.4}$$

for $(x, \xi) \in \Omega'$, $|\xi| \geq (j+1)R$, $j=0, 1, 2, \cdots$. *Then the formal power series* $\exp\{\sum_{j=0}^{\infty} t^j p_j(x, \xi)\}$ *is a formal symbol.*

3. The exponential law. Let $p(x, \xi)$ and $q(x, \xi)$ be symbols defined in Ω. We assume that for each $\Omega' \subset\subset \Omega$ and $h>0$ there is a constant $H>0$ such that

$$\begin{cases} |p(x, \xi)| \leq h|\xi|+H, \\ |q(x, \xi)| \leq h|\xi|+H \end{cases} \tag{3.1}$$

for $(x, \xi) \in \Omega'$.

Let us define a sequence $\{w_j\}_{j=0}^{\infty}$ of symbols of variables $(x, y, \xi, \eta) \in X \times X \times C^n \times C^n \simeq T^*(X \times X)$ by

$$\begin{cases} w_0(x, y, \xi, \eta) = p(x, \xi) + q(y, \eta), \\ w_{j+1} = \dfrac{1}{j+1}\left(\partial_\xi \cdot \partial_y w_j + \sum_{k=0}^{j} \partial_\xi w_k \cdot \partial_y w_{j-k}\right), \qquad j \geq 0. \end{cases} \tag{3.2}$$

Now set $r_j(x, \xi) = w_j(x, x, \xi, \xi)$ for $j=0, 1, 2, \cdots$. Then we have the following

Theorem 3. *The formal sum* $\sum_{j=0}^{\infty} t^j r_j(x, \xi)$ *is a formal symbol which satisfies the condition of Proposition 2 and the following exponential law.*

$$:\exp\{p(x, \xi)\}:\ :\exp\{q(x, \xi)\}: \ = \ :\exp\left\{\sum_{j=0}^{\infty} r_j(x, \xi)\right\}:. \tag{3.3}$$

Remarks. (i) This exponential law is valid for any known class of symbols not only of pseudodifferential operators but also of non-local operators as far as the right-hand side makes sense (cf. [4]–[6]).

(ii) In the case of $n=1$, the first four terms of r_j are as follows.

$$r_0 = p+q,$$
$$r_1 = \partial_\xi p \partial_x q,$$
$$r_2 = \frac{1}{2}\{\partial_\xi^2 p \partial_x^2 q + (\partial_\xi p)^2 \partial_x^2 q + \partial_\xi^2 p (\partial_x q)^2\},$$
$$r_3 = \frac{1}{6}\partial_\xi^3 p \partial_x^3 q + \frac{1}{2}\{\partial_\xi^3 p \partial_x q \partial_x^2 q + \partial_\xi p \partial_\xi^2 p \partial_x^3 q\}$$
$$+\frac{1}{6}\{(\partial_\xi p)^3 \partial_x^3 q + \partial_\xi^3 p (\partial_x q)^3\} + \partial_\xi p \partial_\xi^2 p \partial_x q \partial_x^2 q.$$

In the case of the orders of p and q are smaller than 1, the preceding theorem can be rewritten. Hereafter ρ denotes a real number such that $0 \leq \rho < 1$. Let N be the largest integer such that $(N+1)\rho - N \geq 0$. We assume further that for each $\Omega' \subset\subset \Omega$ there are $h>0$, $H>0$ such that

$$\begin{cases} |p(x, \xi)| \leq h|\xi|^\rho + H, \\ |q(x, \xi)| \leq h|\xi|^\rho + H \end{cases} \tag{3.4}$$

for $(x, \xi) \in \Omega'$. Then we have the following theorem which is natural extension of Theorem 2 in [2], where we assumed $\rho \leq 2/3$.

Theorem 4. *There is a formal symbol* $\sum_{k=0}^{\infty} t^k S_k(x, \xi)$ *which satisfies*

(3.5) $$:\exp\{p(x,\xi)\}: :\exp\{q(x,\xi)\}: = :\exp\left\{\sum_{j=0}^{N} r_j(x,\xi)\right\}\cdot\left\{1+\sum_{k=0}^{\infty} S_k(x,\xi)\right\}:,$$

(3.6) *there is a constant* $C, A>0$ *such that*

$$|S_k(x,\xi)| \leqq CA^k k!^{1-\rho}|\xi|^{-\lambda-k(1-\rho)} \quad \textit{for } (x,\xi)\in\Omega',$$

$k=0, 1, 2, \cdots$. *Here* $-\lambda=(N+2)\rho-(N+1)<0$.

The preceding theorem asserts that the symbol of the composite operator $:e^p: :e^q:$, which is an operator of infinite order, is factorized by $\exp\{\sum_{j=0}^{N} r_j(x,\xi)\}$ and the quotient is a formal symbol of order 0 with principal symbol 1.

4. Invertibility. Theorem 4 yields the following

Theorem 5. *Let* $P=:P(x,\xi):$ *be a holomorphic microlocal operator with symbol* $P(x,\xi)$ *of growth order at most* (ρ) *defined near* $\mathring{x}^*$. *Suppose that* $1/P(x,\xi)$ *is also a symbol of growth order at most* (ρ). *Then* P *is invertible in the ring* $\mathcal{E}^R_{\mathring{x}^*}$.

In the case of $\rho\leq 1/2$ and of $\rho \leq 2/3$, this theorem was given respectively in [1] and in [2].

5. Outline of the proof of Theorem 3. The composite operator $:\exp\{p(x,\xi)\}: :\exp\{q(x,\xi)\}:$ is expressed by $:R(t;x,\xi):$. Here

(5.1) $$R(t;x,\xi)=\sum_{j=0}^{\infty} t^j \sum_{|\alpha|=j}\frac{1}{\alpha!}\partial_\xi^\alpha \exp\{p(x,\xi)\}\cdot\partial_x^\alpha \exp\{q(x,\xi)\}.$$

Set

(5.2) $$\Pi=\sum_{j=0}^{\infty} t^j \sum_{|\alpha|=j}\frac{1}{\alpha!}\partial_\xi^\alpha \exp\{p(x,\xi)\}\cdot\partial_y^\alpha \exp\{q(y,\eta)\}.$$

Then Π satisfies the following differential equation.

(5.3) $$\begin{cases}\partial_t \Pi=\partial_\xi\cdot\partial_y \Pi,\\ \Pi|_{t=0}=\exp\{p(x,\xi)+q(y,\eta)\}.\end{cases}$$

The solution to (5.3) in the space of formal power series in t whose coefficients are differential polynomials of $p(x,\xi)$ and $q(y,\eta)$ is unique. Now we assume that Π has the form

(5.4) $$\Pi=\exp\left\{\sum_{j=0}^{\infty} t^j w_j(x,y,\xi,\eta)\right\}.$$

Then $\{w_j\}$ must satisfy recursion formula (3.2). Since $R(t;x,\xi)=\Pi(t;x,x,\xi,\xi)$, it is clear that $R(t;x,\xi)=\exp\{\sum_{j=0}^{\infty} t^j r_j(x,\xi)\}$ is a formal symbol. However, it is not trivial that $\sum_{j=0}^{\infty} t^j r_j(x,\xi)$ is a formal symbol. To prove it, we have to use the following inequality: There is $C>0$ so that

(5.5) $$\sum_{\mu=0}^{\nu-1}\sum_{k=\mu}^{j-\nu+\mu+1}(k+1)^{k-\mu-2}(j-k+1)^{j-k-\nu+\mu-1}$$

$$\leq C \cdot (j+2)^{j-\nu-1} \qquad \text{for } j=1, 2, 3, \cdots ; \nu=1, 2, \cdots, j.$$

Detailed proof will be published elsewhere.

References

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