38. Sharpness of Parametrices for Strictly Hyperbolic Operators

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(Communicated by Kôsaku YOSIDA, M. J. A., April 12, 1982)

1. Introduction. Let $P(x, D)$ be a linear partial differential operator with C^∞-coefficients defined in R^n and strictly hyperbolic with respect to x_1. Let $\mathrm{E}_k : \mathcal{D}'(Y)\to\mathcal{D}'(\mathrm{R}^n)$ be k-th parametrices, i.e.

$$P(x, D)\mathrm{E}_k\equiv 0, \qquad D_{x_1}^{m-j}\mathrm{E}_k|_{x_1=0}\equiv\delta_{jk}\mathrm{I},$$

where $Y=\{x\in\mathrm{R}^n\,;\, x_1=0\}$ is the initial plane (see e.g. [1]). We want to study the sharpness of distributions $E_k(x, y):=\mathrm{E}_k\delta(x-y)$ here we take $y\in Y$ as parameters. If we take

$$\Lambda=\Lambda(y):=\{(x,\xi)\in T^*\mathrm{R}^n\,;\ (x,\xi) \text{ is on a bicharacteristic strip through some } (y,\eta)\in T^*\mathrm{R}^n \text{ with } P_m(y,\eta)=0\},$$

$$W=W(y):=\pi\Lambda(y),$$

where $\pi: T^*\mathrm{R}^n\to\mathrm{R}^n$ is the natural projection, then we have

$$\operatorname{sing\,supp} E_k(x, y)\subset W(y).$$

Now take a point $x^0\in W$ and a component ω of $\mathrm{R}^n\backslash W$ with $x^0\in\partial\omega$. Then $E_k(x, y)$ is said to be *sharp* at x^0 from ω if there is a neighbourhood V of x^0 and $u\in C^\infty(V)$ such that $E_k(x, y)=u(x)$ on $\omega\cap V$.

Near each point $x^0\in W$, $E_k(x, y)$ can be represented by a finite sum of paired oscillatory integrals $I^\sigma(a, \varphi, x)$, for which L. Gårding [3] discovered a criterion for sharpness. But his arguments and proofs are rather sketchily and, in part, incomplete. Our aim is to clarify the situation and to give a rigorous proofs when $x^0\in W$ is a stable point. Here we use

Definition. $x^0\in W$ is called a stable point if under small perturbations of $\Lambda\subset T^*\mathrm{R}^n$ (as conic Lagrangean manifolds) near $\pi^{-1}(x^0)$, the configurations of W cannot be changed off local diffeomorphisms.

Note that our definition of stability may be considered as a *well posedness* for the problem of sharpness.

If $\pi^{-1}(x^0)\cap\Lambda$ consist of *regular points* (i.e. $N:=\dim T_{\lambda^0}\Lambda\cap T_{\lambda^0}(\text{fibre})=1$ for $\lambda^0\in\pi^{-1}(x^0)\cap\Lambda$), an easy criterion for sharpness are given in [4]. So, in what follows, we shall consider the case when $\pi^{-1}(x^0)\cap\Lambda$ contains *irregular points* (i.e. the case when $N\geqslant 2$).

2. Suppose that $\pi^{-1}(x^0)\cap\Lambda$ consist of stable and irregular points. Then we can prove that, as a germ at x^0, $E_k(x, y)$ can be represented by a finite sum of distributions of the from

$$(*)\qquad G_q^\sigma(x)=\int_V \chi_q^\sigma(\varphi(x,\theta))d\theta,$$

multiplied by smooth functions. Here $q\in \mathrm{Z}/2$, $\chi_q^\sigma(t)=\chi_q(t+i0)+\sigma\chi_q(t-i0)$ and $\sigma=\pm 1$ are determined by the Maslov index. Further $\chi_q(t\pm i0)=\lim_{\varepsilon\downarrow 0}\chi_q(t\pm i\varepsilon)\in \mathcal{D}'(\mathrm{R})$ are defined by boundary values on the real axis of the analytic functions

$$\chi_q(z)=\begin{cases}\Gamma(-q)e^{-\pi i q}z^q, & q\neq 0,1,2,\cdots,\\ z^q(\log z^{-1}+c_q+\pi i)/q\,!, & q=0,1,2,\cdots,\end{cases}$$

defined on $-\pi<\arg z<\pi$, where $c_q=q^{-1}+c_{q-1}$ and $c_0=\Gamma'(1)$. Finally $\varphi(x,\theta)$ is a real valued function with $\dim\theta=N-1$ and

$$\Lambda=\{(x,d_x\varphi(x,\theta))\,;\,\varphi(x,\theta)=0,\ d_\theta\varphi(x,\theta)=0\}\qquad \text{near } \lambda^0,$$

where $\lambda^0\in\pi^{-1}(x^0)\cap\Lambda$ and V is a neighbourhood of θ^0 where $\lambda^0=(x^0,d_x\varphi(x^0,\theta^0))$.

Now we shall study the integral $(*)$. We can assume, without loss of generality, that $(x^0,\theta^0)=(0,0)$. Further we can prove that if $\varphi(x,\theta)$ is stable, there is a local diffeomorphism $(\tilde{x},\tilde{\theta})$ near $(0,0)\in\mathrm{R}^n\times\mathrm{R}^{N-1}$ with $\tilde{x}=\tilde{x}(x)$, $\tilde{\theta}=\tilde{\theta}(x,\theta)$ such that $G_q^\sigma(x)$ can be represented by

$$\int_{\tilde{V}}\chi_q^\sigma(\tilde{\varphi}(\tilde{x},\tilde{\theta}))d\tilde{\theta}$$

multiplied by a smooth function, where $\tilde{\varphi}(\tilde{x},\tilde{\theta})$ is a function of the form

$$(**)\qquad \tilde{\varphi}(\tilde{x},\tilde{\theta})=f_0(\tilde{\theta})+\sum_{j=1}^{k-1}\tilde{x}_j f_j(\tilde{\theta})+\tilde{x}_k.$$

Here $f_j(\tilde{\theta})$ $(j=0,1,\cdots,k-1)$ are certain polynomials of $\tilde{\theta}$ (see e.g. [2]). Thus in what follows we shall assume that $\varphi(x,\theta)$ has the form $(**)$.

In the following we shall consider the case when $q\in\mathrm{Z}$. The case when q is a half integer will be treated similarly. By studying the zeros of the equation $\varphi(x,\theta)+i\varepsilon=0$ for small ε, we have

Lemma 1. *Take a neighbourhood X of $x=0$ small enough. Then, for any fixed $x\in X\backslash W$, there is a small $\varepsilon_0>0$ and a C^∞-vector field $V\ni\theta\mapsto v(x,\theta\,;\varepsilon)\in\mathrm{R}^{N-1}$ such that* (i) $v(x,\theta\,;0)\equiv 0$, (ii) $\varphi(x,\theta+iv(x,\theta\,;\varepsilon))\neq 0$ *for all ε with $0<\varepsilon\leqslant\varepsilon_0$ and* (iii) $d_\theta\varphi(x,\theta+iv(x,\theta\,;\varepsilon))\cdot((dv(x,\theta\,;\varepsilon))/d\varepsilon)|_{\varepsilon=0}>0$.

Further we have

Lemma 2. *For any fixed $x\in X\backslash W$, there is a neighbourhood U of x such that* (i) $\varphi(y,\theta+iv(x,\theta\,;\varepsilon))\neq 0$ *and* (ii) $d_\theta\varphi(y,\theta+iv(x,\theta\,;\varepsilon))\cdot((dv(x,\theta\,;\varepsilon))/d\varepsilon)|_{\varepsilon=0}>0$ *for all $y\in U$.*

By Lemma 1, if $x\in X\backslash W$, we can represent $G_q^\sigma(x)$ as

$$G_q^\sigma(x)=\int_{\gamma_\sigma(x)}\chi_q^\sigma(\varphi(x,\theta))d\theta,$$

where $\gamma_\sigma(x)$ is an $(N-1)$-chain with natural orientation in a complex neighbourhood $\hat{V}\subset\mathrm{C}^{N-1}$ of $V\subset\mathrm{R}^{N-1}$ determined by $v(x,\theta\,;\varepsilon)$ and σ. Further, by Lemma 2, we have

$$G_q^\sigma(y)=\int_{\gamma_\sigma(x)}\chi_q^\sigma(\varphi(y,\theta))d\theta\qquad\text{for all } y\in U.$$

Especially we have that $G_q^\sigma(x)$ is analytic in $X\backslash W$. Using this expression, we can make a criterion for sharpness of $G_q^\sigma(x)$ in terms of the chain $\gamma_\sigma(x)$ or its homology class.

3. Let us fix a complex neighbourhood $\hat{V}\subset \mathrm{C}^{N-1}$ of $V\subset \mathrm{R}^{N-1}$ and take $X\subset \mathrm{R}^k$ as a small neighbourhood of the origin. Note that, though X may depend on V, (*) defines the same germs at the origin modulo analytic functions as long as V contains the origin.

For fixed $x\in X$, we write

$$\mathrm{V}_x:=\hat{V}\backslash\{\theta;\varphi(x,\theta)=0\},\qquad \delta\mathrm{V}_x:=\partial\hat{V}\backslash\{\theta;\varphi(x,\theta)=0\}.$$

Then for each $x\in X\backslash W$, the chain $\gamma_\sigma(x)$ determines an $(N-1)$-th relative homology class $\alpha(x;\sigma):=[\gamma_\sigma(x)]\in H_{N-1}(\mathrm{V}_x,\delta\mathrm{V}_x)$.

Next we take a point $x^0\in W$ and a component ω of $X\backslash W$. Further take a smooth path $\ell=\{x_t;0\leqslant t\leqslant 1\}$ in $\bar{\omega}$ with an end point $x^0=x_t|_{t=0}$ such that $\ell\cap\partial\omega=\{x^0\}$. Then we can formulate our criterion for sharpness.

Theorem 1. *$G_q^\sigma(x)$ is sharp at x^0 from ω if there is a relative cycle γ in X such that* (i) $[\gamma]\in H_{N-1}(\mathrm{V}_{x^0},\delta\mathrm{V}_{x^0})$ *and* (ii) $[\gamma]=[\gamma_\sigma(x_t)]$ *in* $H_{N-1}(\mathrm{V}_{x_t},\delta\mathrm{V}_{x_t})$ *for every sufficiently small* $t>0$.

We call the condition (i) and (ii) of Theorem 1 the *local Petrowsky condition.*

Now we return to the problem for the distributions $E_k(x,y)$ themselves. Then we have

Theorem 2. *Take $x^0\in W$ and a curve ℓ with an end point x^0 in a component ω of $\mathrm{R}^n\backslash W$. Suppose that all the points in $\pi^{-1}(x^0)\cap\Lambda$ are stable points. Then there associate 1-parameter families of relative homology groups as above such that the local Petrowsky condition implies the sharpness of $E_k(x,y)$ at x^0 from ω.*

Next, we write

$$Z:=(\ell\times\hat{V})\backslash\{(x,\theta);\varphi(x,\theta)=0\},\qquad \delta Z:=(\ell\times\partial\hat{V})\backslash\{(x,\theta);\varphi(x,\theta)=0\}.$$

Let $\iota:(\mathrm{V}_x,\delta\mathrm{V}_x)\hookrightarrow(Z,\delta Z)$ be the inclusion mappings and let

$$\iota_*:H_{N-1}(\mathrm{V}_x,\delta\mathrm{V}_x)\ni\alpha\longmapsto\alpha_*\in H_{N-1}(Z,\delta Z)$$

be mappings induced by ι. If we can clarify the structure of the mappings ι_*, we can restate the theorems (or local Petrowsky condition) in more explicit way. For example, suppose $\varphi(x,\theta)$ is an A_m-type function; i.e.

$$\varphi(x,\theta)=\theta^{m+1}+x_1\theta^{m-1}+\cdots+x_{m-1}\theta+x_m.$$

Then $\alpha(x;\sigma)$ determine the same homology class in $H_1(Z,\delta Z)$ as long as x belongs to the same component ω. We shall write this class by $\alpha_*(\omega;\sigma)\in H_1(Z,\delta Z)$. Then we have

Theorem 3. *Let $\varphi(x,\theta)$ be an A_m-type function. Then $G_q^\sigma(x)$ is sharp at x^0 from ω if and only if there is a chain $\beta\in H_1(\mathrm{V}_{x^0},\delta\mathrm{V}_{x^0})$ such that $\alpha_*(\omega;\sigma)=\iota_*\beta$.*

For the case when q is a half integer, a similar arguments are possible. In doing so, however, we have to consider instead of V_x etc., the double covering of V_x etc., branched at $\varphi(x, \theta)=0$. In this case, there are delicate problems in selection of branches of the cycles. The details will appear elsewhere [6]. See also [5] for explicit calculations when $\varphi(x, \theta)$ is A_m-type.

References

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