37. *Potential Theory and Eigenvalues of the Laplacian*

By Shin OZAWA
Department of Mathematics, University of Tokyo

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§1. **Introduction.** We consider a bounded domain Ω in $\mathbf{R}^3$ with C^2 boundary γ. We fix a point w in Ω. Let D be an open neighbourhood of the origin. Let $D(\varepsilon, w)$ be the set defined by $D(\varepsilon, w)=\{x\in\mathbf{R}^3;\ \varepsilon^{-1}(x-w)\in D\}$. We put $\Omega(\varepsilon)=\Omega\backslash\overline{D(\varepsilon, w)}$. Let $0>m_1(\varepsilon)\geq m_2(\varepsilon)\geq\cdots$ be the eigenvalues of the Laplacian in $\Omega(\varepsilon)$ under the Dirichlet condition on $\partial\Omega(\varepsilon)$. Let $0>m_1\geq m_2\geq\cdots$ be the eigenvalues of the Laplacian in Ω under the Dirichlet condition on γ. We arrange them repeatedly according to their multiplicities.

We proposed the following problem in Ozawa [1].

Problem. *Describe the precise asymptotic behaviour of $m_j(\varepsilon)$ as ε tends to zero.*

And the author conjectured in [1] the following

Conjecture. *Fix j. Assume that the multiplicity of m_j is one, then there exists a constant $c(D)$ such that*

$$m_j(\varepsilon)-m_j=-4\pi c(D)\varepsilon\varphi_j(w)^2+0(\varepsilon^{3/2}) \tag{1.1}$$

holds as ε tends to zero. Here $\varphi_j(x)$ is the normalized eigenfunction of the Laplacian associated with m_j.

In this note we give an answer to the above problem. We have the following

Theorem 1. *Under the same assumption as above,* (1.1) *holds and $c(D)$ is the electrostatic capacity* cap (D) *of the set D. Moreover,*

$$m_j(\varepsilon)-m_j+4\pi\,\mathrm{cap}\,(D)\varepsilon\varphi_j(w)^2=0(\varepsilon^{2-s}) \tag{1.2}$$

holds for an arbitrary fixed $s>0$ as ε tends to zero.

Remark. We define cap (D) by

$$\mathrm{cap}\,(D)=-(4\pi)^{-1}\int_{\partial D}\frac{\partial u}{\partial\nu}\,d\sigma, \tag{1.3}$$

where u is the unique solution of

$$\Delta u=0\quad\text{in } D^c,\quad u|_{\partial D}=1,\quad \lim_{|x|\to\infty}u(x)=0. \tag{1.4}$$

Here $d\sigma$ is the surface element of ∂D.

The above Theorem 1 is a generalization of Theorem 2 in Ozawa [2]. The work in this paper was heavily inspired by the paper Papanicolau-Varadhan [5] in which "many holes problem" was studied.

In §2, we give an outline of the proof of Theorem 1. Details of this paper will be given elsewhere.

§ 2. Outline of the proof of Theorem 1. It should be remarked that $\Omega(\varepsilon)$ may not be connected. But for the sake of simplicity, we study the case where $\Omega(\varepsilon)$ is connected.

Let $G(x, y)$ (resp. $G_\varepsilon(x, y)$) be the Green function of the Laplacian in Ω (resp. $\Omega(\varepsilon)$) under the Dirichlet condition on γ (resp. $\partial\Omega(\varepsilon)$). We put

$$G^{(2)}(x, y)=\int_{\Omega} G(x, z)G(z, y)dz$$

for any $x,\ y \in \Omega$. And we put

$$G_\varepsilon^{(2)}(x, y)=\int_{\Omega(\varepsilon)} G_\varepsilon(x, z)G_\varepsilon(z, y)dz$$

for any $x,\ y \in \Omega(\varepsilon)$.

Let $\tilde{u}(x) \in C^2(R^3)$ be an extension of u. Put $\Delta\tilde{u}(x)=\rho(x)$ for any $x \in \mathrm{R}^3$. And we put $\rho_\varepsilon(x)=\rho(\varepsilon^{-1}(x-w))$. Then $\rho_\varepsilon(x)=0$ on $\overline{\Omega(\varepsilon)}$.

We introduce the integral kernal $p_\varepsilon(x, y)$ given by the following:

$$p_\varepsilon(x, y)=G^{(2)}(x, y)-\varepsilon^{-2}\sum_{h=1}^{2}\int_{D(\varepsilon,w)} G^{(3-h)}(x, v)G^{(h)}(y, v)\rho_\varepsilon(v)dv \tag{2.1}$$

for any $x,\ y \in \Omega$. Here $G^{(1)}(x, y)=G(x, y)$.

We put

$$(\boldsymbol{Q}_\varepsilon f)(x)=\int_{\Omega(\varepsilon)} (G_\varepsilon^{(2)}(x, y)-p_\varepsilon(x, y))f(y)dy.$$

Then we have the following

Proposition 1. *There exists a constant C_s independent of ε such that*

$$\|\boldsymbol{Q}_\varepsilon f\|_{L^2(\Omega(\varepsilon))}\leq C_s\varepsilon^{2-s}\|f\|_{L^2(\Omega(\varepsilon))} \qquad (s>0)$$

holds. Here C_s may depend on $s>0$.

We use L^p $(1<p<\infty)$ function spaces to get Proposition 1.

Since $p_\varepsilon(x, y)$ is written explicitly by the original Green function $G(x, y)$, we can construct an approximate eigenvalue of $\boldsymbol{P}_\varepsilon$ defined by $\boldsymbol{P}_\varepsilon f(x)=\int_{\Omega(\varepsilon)} p_\varepsilon(x, y)f(y)dy$. More explicitly, we have the following

Proposition 2. *Let m_j be as before. Then there exists at least one eigenvalue $g_j(\varepsilon)$ of $\boldsymbol{P}_\varepsilon$ satisfying*

$$g_j(\varepsilon)-m_j^{-2}=-2m_j^{-3}\varepsilon^{-2}\int_{D(\varepsilon,w)} \varphi_j(v)^2\rho_\varepsilon(v)dv+0(\varepsilon^2). \tag{2.2}$$

On the other hand, we have

$$\begin{aligned}\varepsilon^{-2}&\int_{D(\varepsilon,w)} \varphi_j(v)^2\rho_\varepsilon(v)dv \\ &=\varepsilon^{-2}\varphi_j(w)^2\int_{D(\varepsilon,w)} \rho_\varepsilon(v)dv+0(\varepsilon^2) \\ &=-4\pi\varepsilon \operatorname{cap}(D)\varphi_j(w)^2+0(\varepsilon^2).\end{aligned} \tag{2.3}$$

Therefore there exists at least one eigenvalue $g_j(\varepsilon)$ of $\boldsymbol{G}_\varepsilon^2$ satisfying

$$g_j(\varepsilon)-m_j^{-2}=-2m_j^{-3}(-4\pi\varepsilon \operatorname{cap}(D)\varphi_j(w)^2)+0(\varepsilon^{2-s}) \qquad (s>0)$$

as ε tends to zero. Here $\boldsymbol{G}_\varepsilon^2$ is the square of the Green operator $\boldsymbol{G}_\varepsilon$

with the integral kernel $G_\varepsilon(x, y)$. We know $g_j(\varepsilon)=m_j(\varepsilon)^{-2}$ by the result of Rauch-Taylor [6]. See also [2]. Thus we have Theorem 1.

References

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