4. A Calculus of the Gauss-Manin System of Type A_l. I

The Residual Representation

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(Communicated by Kôsaku YOSIDA, M. J. A., Jan. 12, 1982)

The present note is the former half of our article titled "A calculus of the Gauss-Manin system of type A_l". For the latter half, see [4].

0. Introduction. Let $F=x^l+t_2x^{l-2}+\cdots+t_l$ be the versal deformation of the isolated singularity $x^l=0$ of type A_{l-1} and consider the integral

$$u(t)=\int \delta(F(x,t))dx, \qquad t=(t_2,\cdots,t_l). \tag{0.1}$$

In the present article, we propose two types of explicit representations of the Gauss-Manin system H_F of type A_{l-1} i.e. the system of micro-differential equations associated with the integral (0.1). (Theorems 1 and 5.) In Theorem 1, we give a matricial representation of the Gauss-Manin system H_F for the *flat basis*, which we call the *residual representation*. (See no. 2.) In Theorem 5, we propose the *Hamiltonian representation* of H_F in terms of the *flat coordinates* introduced by K. Saito, T. Yano and J. Sekiguchi [2]. (See no. 4.) Our construction of the two representations is based on an interesting connection between the flat coordinates of type A_{l-1} and the *fractional power* $F^{1/l}$ of F. (See nos. 1 and 3.) The Hamiltonian representation allows us to calculate explicitly the quantized contact transformation which reduces the Gauss-Manin system H_F to a standard form (Theorem 6). The details of the following arguments will be published elsewhere.

1. The flat basis. Let R be the polynomial ring $C[s_2,s_3,s_4,\cdots]$ of countably many variables $s_2,s_3,s_4,\cdots$ and $R((x^{-1}))$ the ring $R[[x^{-1}]][x]$ of formal Laurent series in x^{-1} with coefficients in R. By the definition, each element ϕ of $R((x^{-1}))$ is written as a formal sum

$$\phi=\sum_{i=0}^{\infty}\phi_i x^{m-i}, \tag{1.1}$$

where m is an integer and $\phi_i\in R$ for each $i\in N$. Such a ϕ is said to be of degree m if $\phi_0\neq 0$. We denote by $\mathrm{Res}_x(\phi)$ the coefficient ϕ_{m+1} of x^{-1} and by $(\phi)_+$ the polynomial part of ϕ:

$$(\phi)_+=\sum_{i=0}^{m}\phi_i x^{m-i}. \tag{1.2}$$

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The residue symbol Res_x is characterized as the unique R-homomorphism $R((x^{-1}))\to R$ satisfying the following conditions:

i) $\mathrm{Res}_x(\partial_x(\phi))=0$ for any $\phi\in R((x^{-1}))$ and

ii) $\mathrm{Res}_x(\partial_x(\phi)/\phi)=\deg_x(\phi)$ if $\phi\in R((x^{-1}))$ is invertible, where $\partial_x=\partial/\partial x$.

For the variables $s_2, s_3, s_4, \cdots$ in R, we set

(1.3) $$f=x+\sum_{i=2}^{\infty}s_i x^{1-i}.$$

Moreover, we define two sequences $(F_k)_{k\in N}$ and $(e_k)_{k\in N}$ of monic polynomials in $R[x]$ by

(1.4) $$F_k=(f^k)_+ \text{ and } e_k=(\partial_x(f)f^k)_+.$$

Proposition 1. *Let $(F_k)_{k\in N}$ be as above. Then we have*

(1.5) $$\deg_x(lF_l\partial_x(F_k)-kF_k\partial_x(F_l))\leq l-2 \text{ for } k\leq l$$

and

(1.6) $$\deg_x(\partial_{s_i}(F_l)\partial_x(F_k)-\partial_{s_i}(F_k)\partial_x(F_l))\leq l-2 \text{ for } k\leq l.$$

Proposition 2 (Flatness of $(e_k)_{k\in N}$). *Let $(e_k)_{k\in N}$ be as in* (1.4). *Then, for any integers i, j and k with $0\leq i, j\leq k$, we have*

$$\mathrm{Res}_x(e_ie_j/e_k)=\begin{cases}1 & \text{if } i+j-k=-1,\\ 0 & \text{if } i+j-k\neq -1.\end{cases}$$

In view of Proposition 2, the sequence $(e_k)_{k\in N}$ will be called the *flat basis* for $R[x]$.

Now let $F=x^l+t_2x^{l-2}+\cdots+t_l$ be the versal deformation of the isolated singularity $x^l=0$ of type A_{l-1}. Let R_l be the polynomial ring $C[t_2,\cdots,t_l]$ of $l-1$ variables $t_2,\cdots,t_l$ and $R_l((x^{-1}))$ the ring of formal Laurent series in x^{-1} with coefficients in R_l. Then we can take the fractional power $F^{1/l}$ of F in $R_l((x^{-1}))$:

(1.7) $$F^{1/l}=\sum_{i=0}^{\infty}(1+t(u))_i^{1/l}x^{1-i},$$

where we set

$$t(u)=\sum_{i=2}^{l}t_iu^i$$

for an indeterminate u and $(1+t(u))_i^{1/l}$ stands for the coefficient of u^i in the Taylor expansion of $(1+t(u))^{1/l}$. Noting this, we define a ring-homomorphism $\rho_l: R\to R_l$ by

$$\rho_l(s_i)=(1+t(u))_i^{1/l} \qquad \text{for } i=2, 3, \cdots.$$

Then the kernel of ρ_l is the ideal J_l of R generated by the polynomials $(1+s(u))_j^l (j>l)$, where $s(u)=\sum_{i=2}^{\infty}s_iu^i$. The isomorphisms of rings

$$R/J_l\xrightarrow{\sim}R_l \quad\text{and}\quad R/J_l((x^{-1}))\xrightarrow{\sim}R_l((x^{-1}))$$

will be called the homomorphisms of *l-reduction*. With this identification, the l-reduction of F_k, e_k or s_i will be denoted by the same symbol. Then we have

(1.8) $$F_k=(F^{k/l})_+ \quad\text{and}\quad e_k=\frac{1}{k+1}(\partial_x(F^{(k+1)/l}))_+$$

in $R_l[x]$.

2. The Gauss-Manin system of type A_{l-1}.

Fix an integer $l \geq 2$ and consider the versal deformation

$$F = x^l + t_2 x^{l-2} + \cdots + t_l$$

of type A_{l-1}. Let $(y_2, \cdots, y_l)$ be a coordinate system for the space of parameters $(t_2, \cdots t_l)$ such that

i) y_j is a polynomial without constant term in $(t_2, \cdots, t_l)$ for $j = 2, \cdots, l$, and

ii) $\partial_{t_i}(y_i) = 1$ and $\partial_{t_j}(y_i) = 0$ for $i < j$.

We recall the Gauss-Manin system $\underline{H}_F$ for F i.e. the system of micro-differential equations associated with the integral of the delta function $\delta(F)$. (For the details, see F. Pham [1].)

Let $Z = \boldsymbol{C}^l$ be the complex affine l-space with coordinates $(x, y_2, \cdots, y_l)$ and $S = \boldsymbol{C}^{l-1}$ the complex affine $(l-1)$-space with coordinates $(y_2, \cdots, y_l)$. Then the sheaf $\mathcal{C}_{[F]}$ over the cotangent bundle T^*Z is the micro-localization of the sheaf $\mathscr{B}_{[F]}$ of algebraic hyperfunctions with supports in $\{F = 0\}$ defined by

$$\mathscr{B}_{[F]} = \mathcal{O}_Z[F^{-1}]/\mathcal{O}_Z,$$

where $\mathcal{O}_Z$ is the sheaf of holomorphic functions over Z. The modulo class of $-(1/2\pi i)\cdot 1/\mathrm{F}$ in $\mathscr{B}_{[F]}$ or $\mathcal{C}_{[F]}$ is denoted by $\delta(F)$. Let ρ and $\tilde{\omega}$ be the canonical morphisms

$$'T^*Z \xleftarrow{\rho} Z \underset{S}{\times} {'T^*S} \xrightarrow{\tilde{\omega}} {'T^*S}$$

and consider the relative De Rham complex $\underline{\mathrm{DR}}^{\boldsymbol{\cdot}}_{Z/S}(\mathcal{C}_{[F]})$ with coefficients in $\mathcal{C}_{[F]}$. Then we set

$$\underline{H}_F = \underline{\mathrm{H}}^1(\tilde{\omega}_* \rho^{-1}(\underline{\mathrm{DR}}^{\boldsymbol{\cdot}}_{Z/S}(\mathcal{C}_{[F]})) = \int^0_{S \leftarrow Z} \mathcal{C}_{[F]},$$

which is the integration of $\mathcal{C}_{[F]}$ along the fibres of the canonical projection $Z \to S$. The sheaf $\underline{H}_F$ over $'T^*S$ has a natural structure of a left Module over the Ring $\mathcal{E}_S$ of micro-differential operators over S. Hereafter, we denote by H_F the stalk $\underline{H}_{F,(0,dy_l)}$ of $\underline{H}_F$ and call H_F the Gauss-Manin system associated with F. With a canonical good filtration $(H_F^{(k)})_{k \in Z}$, H_F is a simple holonomic system with generator

$$u = \int \delta(F)dx \in H_F^{(0)}.$$

We remark that $H_F^{(0)}$ is a free module of rank $l-1$ over the ring $\boldsymbol{C}\{y_2, \cdots, y_{l-1}\}\{\{\partial_y^{-1}\}\}$.

Now we take the sequence $e_0, \cdots, e_{l-2}$ of monic polynomials in $R_l[x]$ defined by (1.8) and set

$$u_i = \int e_i \delta(F)dx \qquad \text{for } i = 0, \cdots, l-2.$$

Then $u_0, \cdots, u_{l-2}$ form a free basis of $H_F^{(0)}$ over the ring $\boldsymbol{C}\{y_2, \cdots, y_{l-1}\}\{\{\partial_y^{-1}\}\}$, which we call the *flat basis* for the Gauss-Manin system H_F.

The following theorem gives a "residual" representation of the Gauss-Manin system H_F as a system of micro-differential equations for the vector $\vec{u}={}^t(u_0, \cdots, u_{l-2})$ of unknown functions.

Theorem 1. *Let $u_0, \cdots, u_{l-2}$ be the flat basis for H_F. Then the Gauss-Manin system H_F of type A_{l-1} is given by*

$$(2.1)\qquad \begin{cases} y_l\vec{u}=A_0\vec{u}+A_1\partial_{y_l}^{-1}\vec{u} & \text{and} \\ \partial_{y_k}\partial_{y_l}^{-1}\vec{u}=B^{(k)}\vec{u} & \text{for } k=2, \cdots, l-1. \end{cases}$$

Here A_1 is the diagonal matrix of size $l-1$ whose diagonal components are $(1/l, 2/l, \cdots, (l-1)/l)$. A_0 and $B^{(k)}$ $(k=2, \cdots, l-1)$ are determined by the following residual representations:

$$A_0=(a_{ij})_{0\le i,j\le l-2}\in M(l-1, \boldsymbol{C}[y_2, \cdots, y_{l-1}]),$$

where

$$(2.2)\qquad a_{ij}=l\operatorname{Res}_x(e_ie_{l-2-j}(y_l-F)/\partial_x(F))$$

and

$$B^{(k)}=(b_{ij}^{(k)})_{0\le i,j\le l-2}\in M(l-1, \boldsymbol{C}[y_2, \cdots, y_{l-1}]),$$

where

$$(2.3)\qquad b_{ij}^{(k)}=l\operatorname{Res}_x(e_ie_{l-2-j}\partial_{y_k}(F)/\partial_x(F)).$$

Theorem 1 is a consequence of Propositions 1 and 2 in no. 1.

By the compatibility condition of the system (2.1), we have

Proposition 3. (i) *$[B^{(k)}, A_0]=0$ for $k=2, \cdots, l-1$ and*

(ii) *$[B^{(k)}, A_1]-B^{(k)}=\partial_{y_k}(A_0)$ for $k=2, \cdots, l-1$.*

References

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[4] S. Ishiura and M. Noumi: A calculus of the Gauss-Manin system of type A_l. II (to appear in Proc. Japan Acad., **58**A(2)).