36. Recurrences Defining Rational Approximations to Irrational Numbers

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§ 0. We study diophantine approximations to irrational numbers that are values of the logarithmic (or inverse trigonometric) function using linear recurrences that define nominators and denominators of rational approximations. Asymptotics of solutions to these recurrences then define the measure of the diophantine approximation. Most attention is devoted to two particular numbers, ln 2 and $\pi/\sqrt{3}$, for which new "dense" families of rational approximations are presented.

§ 1. **Lemma 1.1.** *Let θ be a complex number and let us suppose we have the system of linear forms*

$$R_N=\sum_{i=0}^{m-1}\theta^i\cdot P_{i,N}$$

with rational integer coefficients $P_{i,N}$ satisfying the following properties:

i) *for $N\to\infty$ and $|\alpha|<1$, $|R_N|\sim\alpha^N$ or $\log|R_N|/N\to\log\alpha$ as $N\to\infty$;*

ii) *for any $i=0,1,\cdots,m-1$, $\log|P_{i,N}|/N\leq\log\beta$ as $N\to\infty$.*

Then for any rational integers p, q we have

$$|\theta-p/q|>|q|^{-\ae-\varepsilon}\ for\ |q|\geq q_0(\varepsilon)$$

and any $\varepsilon>0$. Here $\ae=(m-1)\{-\log\beta/\log\alpha+1\}$.

If some family of rational approximation P_n/Q_n to θ is found, then the determination of sizes of P_n and Q_n like in ii), is called an "arithmetic" asymptotic, while the error of approximation, as in i), is called an "analytic" asymptotic of rational approximation. The essential in the correct determination of "analytic" and "arithmetic" asymptotics of rational approximations is to establish the (linear) recurrence formulas connecting successive P_n and Q_n:

Lemma 1.2. *Let*

$$\sum_{i=0}^{m}a_i(n)X_{n+i}=0 \tag{1.3}$$

be a linear recurrence with coefficients depending on n such that $a_i(n)\to a_i$ when $n\to\infty$. Let the roots of the "limit" characteristic polynomial $\sum_{i=0}^{m}a_i\lambda^i=0$ be distinct in absolute values: $|\lambda_1|>\cdots>|\lambda_m|$.

Then there are m solutions $X_n^{(j)}$: $j=1,\cdots,m$ of (1.3) such that

$$\log|X_n^{(j)}|\sim n\log|\lambda_1|:\ j=1,\cdots,m$$

as $n\to\infty$ and there is only one (up to a scalar multiplier) solution $\overline{X}_n$ of (1.3) such that

$$\log|\overline{X}_n|\sim n\log|\lambda_m|.$$

§ 2. One of the situations, in which we know both the "arithmetic" and "analytic" asymptotics, is the case when the numbers under consideration are values of (generalized) hypergeometric functions. The corresponding family of rational approximations is a specialization of a system of Padé approximations to (generalized) hypergeometric functions [5], [6].

Definition 2.1. Let $f_1(x),\cdots,f_n(x)$ be functions analytic at $x=0$ and let $m_1,\cdots,m_n$ be non-negative integers (called weights). Then polynomials $P_1(x),\cdots,P_n(x)$ of degrees at most $m_1,\cdots,m_n$ are called Padé approximants to $f_1(x),\cdots,f_n(x)$ if $R(x)=\sum_{i=1}^n P_i(x)f_i(x)$ has a zero at $x=0$ of order $\geq\sum_{i=1}^n(m_i+1)-1$.

The function $R(x)$ is called the remainder function. The asymptotic of Padé approximations for logarithmic and similar functions is given in the following

Theorem 2.2. *Let $w_1,\cdots,w_n$ be distinct* (mod Z) *complex numbers. Let $f_1(x),\cdots,f_n(x)$ be one of the following system of functions:*

i) $f_i(x)=(1-x)^{w_i}: i=1,\cdots,n;$

ii) $f_i(x)={}_2F_1(1;w_i;c;x)\ i=1,\cdots,n;$

iii) $f_i(x)=\log(1-x)^{i-1}: i=1,\cdots,n.$

Let $R(x)=\sum_{i=1}^n P_i(x)f_i(x)$ be the remainder function in the Padé approximation to $f_1(x),\cdots,f_n(x)$ with weights $m_1,\cdots,m_n$ at $x=0$. Let $m=M+m_i^0$ and $m_i^0/M\to 0$ as $M\to\infty: i=1,\cdots,n$.

Then the asymptotics of $|R(x)|$ and $|P_i(x)|$ are determined everywhere in **C** *using the following notations:*

$$r_n^-(x)=\min\{|1-\zeta_n^j\sqrt[n]{1-x}|: j=0,\cdots,n-1\},$$
$$r_n^+(x)=\max\{|1-\zeta_n^j\sqrt[n]{1-x}|: j=0,\cdots,n-1\}.$$

Then for any $x\neq 0,1,\infty$ where $r_n^-(x)<r_n^+(x)$ we have

$$|R(x)|\cong r_n^-(x)^{nM}\Big(1+0\Big(\frac{1}{M}\Big)\Big);\qquad |P_i(x)|\cong r_n^+(x)^{nM}\Big(1+0\Big(\frac{1}{M}\Big)\Big):$$

$i=1,\cdots,n$ as $M\to\infty$. Here $\zeta_n^j=\exp((2\pi\sqrt{-1}\cdot j)/n)$.

To find denominators of Padé approximants and to determine "arithmetic" asymptotics, the recurrences relating Padé approximants with contiguous weights should be analyzed. These recurrences for systems of functions satisfying Fuchsian linear differential equations, are called contiguous relations following Riemann [1].

The recurrences that are consequences of Gauss contiguous relations between ${}_2F_1$ functions can be presented in the form:

$$F(m+1,l,k;z)=F(m,l,k-1;z)+zF(m,l,k;z);$$
$$F(m,l+1,k;z)=F(m,l,k-1;z)+(z-1)F(m,l,k;z).$$

Specification of initial conditions $F(1,1,k;z)$ gives us $P_n(z)$, $Q_n(z)$, $R_n(z)$ in the Padé approximation problem for $\ln(1-(1/z))$:

$$P_n(z)\ \ln\left(1-\frac{1}{z}\right)+Q_n(z)=R_n(z)$$

where $R_n(z)=0(z^{-n-1})$ as $|z|\to\infty$, $P_n(z)$ and $Q_n(z)$ are polynomials of degrees n and $n-1$ respectively.

i) If $F_1(1,1,k;z)=(1/(k-2))\{(-z)^{2-k}-(1-z)^{2-k}\}$ for $k\neq 2$ and $F_1(1,1,2;z)=\ln(1-(1/z))$, $R_n(z)\overset{\text{def}}{=}F_1(n+1,\ n+1,\ n+2;z)$;

ii) If $F_2(1,1,k;z)=\delta_{k2}$, then $P_n(z)\overset{\text{def}}{=}F_2(n+1,n+1,n+2,z)$;

iii) If $F_3(1,1,k;\ z)=(1/(k-2))\{(-z)^{2-k}-(1-z)^{2-k}\}$, $F_3(1,1,2;z)=0$, then $Q_n(z)\overset{\text{def}}{=}F_3(n+1,n+1,n+2;z)$.

These recurrences are usually substituted by a single three-term recurrence

(2.3) $$(n+1)X_{n+1}-(2n+1)(z-2)X_n+nz^2X_{n-1}=0$$

satisfied by $X_n=P_n$, Q_n or R_n.

iv) Coefficients of $P_n(z)$ are rational integers; and

v) coefficients of $Q_n(z)$ are rational numbers with the common denominator dividing the least common multiplier of $1,\cdots,n$, denoted by $lcm\ \{1,\cdots,n\}$ (i.e. growing not faster than $e^{(1+0(1))n}$ as $n\to\infty$).

We start with the number ln 2, which corresponds in the above mentioned scheme to $z=-1$. Lemma 1.1 immediately gives us as in [2]:

$$|q\ln 2-p|>q^{-3.660137409\cdots-\varepsilon}$$

for the rational integers p, q, provided that $|q|\geq q_1(\varepsilon)$ for any $\varepsilon>0$. Similarly, Padé approximations to logarithmic function at points of Gaussian field $\boldsymbol{Q}(i)$ give us measure of diophantine approximation to $\pi/\sqrt{3}$ [2], [8]:

$$|q\pi/\sqrt{3}-p|>|q|^{-7.30998634\cdots-\varepsilon}$$

for rational integers p, q, provided that $|q|\geq q_2(\varepsilon)$ for any $\varepsilon>0$.

The exponent $7.309\cdots$ in the measure of irrationality of $\pi/\sqrt{3}$ is connected with the following nice three-term linear recurrence:

$$\left.\begin{array}{l} n(2n+1)(4n-3)X_{n+1} \\ \quad+\{7\cdot 16n^3-7\cdot 12n^2-6n+5\}X_n \\ \quad+(4n+1)(2n-1)(n-1)X_{n-1}=0 \end{array}\right| \begin{array}{l} \text{as } n\to\infty \\ \quad x^2+7x+1=0, \quad \text{where } X_n\sim x^n \\ \text{as } n\to\infty. \end{array}$$

Then there are two solutions p_n and q_n of this recurrence such that $q_n\in \boldsymbol{Z}$ for all n and $p_n\cdot lcm\{1,\cdots,2n\}\in \boldsymbol{Z}$ for all n; by Poincaré theorem

$$|p_n|\sim(2+\sqrt{3})^{2n},\qquad |q_n|\sim(2+\sqrt{3})^{2n}.$$

Then

$$\left|q_n\frac{\pi}{\sqrt{3}}-p_n\right|\sim(2-\sqrt{3})^{2n}.$$

The expression for p_n, q_n are:

$$q_n=2^{1-2n}\sum_{m=0}^{n-1}\binom{2n-1}{m}\binom{4n-2m-2}{2n-1}3^{n-m} \quad \text{and}$$

$$p_n=\sum_{m=0}^{2n-1}\frac{(2n+m-1)!}{(m!)^2\cdot(2n-1-m)!}\sigma(m)\rho^m-\sigma(2n-1)q_n$$

for $\sigma(m)=1+(1/2)+\cdots+(1/m)$, $\rho=\exp(2\pi i/3)$.

Similarly Padé approximants to binomial functions of i) in Theorem 2.2 provide the effectivization of Thue-Siegel theorem on diophantine approximations to certain classes of algebraic numbers. From the results of [3] it follows that for all algebraic numbers in the following fields, for example, an effectivization of Thue-Siegel theorem is valid:

$$\boldsymbol{Q}(\sqrt[3]{2}),\quad \boldsymbol{Q}(\sqrt[3]{3}),\quad \boldsymbol{Q}(\sqrt[3]{5}),\quad \boldsymbol{Q}(\sqrt[3]{6}).$$

For example, for any irrational number $\alpha\in\boldsymbol{Q}(\sqrt{2})$ we have

$$|\alpha-p/q|>c(\alpha)\cdot|q|^{-2.429\cdots}$$

for arbitrary rational integers p, q. Here the constant $c(\alpha)>0$ depends effectively on $H(\alpha)$ only.

§ 3. Here we present recurrences and their solutions that provide "dense" sequences of rational approximations to ln 2 and $\pi/\sqrt{3}$ with "density constants" better than rational approximations given by the Padé approximation to the logarithmic function. The new, better measure of irrationality of ln 2 is based on a new set of contiguous relations, reflecting the presence of apparent singularities:

$$\begin{aligned}&G(m+1,n,k;z)=G(m,n,k-2;z)+(2z-1)G(m,n,k-1;z)\\ &\qquad\qquad +(z^2-z)G(m,n,k;z);\\ &G(m,n+1,k;z)=G(m,n,k-2;z)+G(m,n,k;z)(z-z^2).\end{aligned}\tag{3.1}$$

There are two kinds of initial conditions that determine sequences P_n and Q_n:

i) $G_1(1,1,k;z)=\delta_{k2}$; $P_n\overset{\text{def}}{=}G_1(N_1,N_2,N_3;-1)$, $N_1=[0.88n]$, $N_2=[0.12n]$, $N_3=n$.

ii) $G_2(1,1,k;z)=(1/(k-2))\{(1-z)^{2-k}-(-z)^{2-k}\}$ for $k\neq 2$, $G_2(1,1,2;z)=0$. Then $Q_n\overset{\text{def}}{=}G_2(N_1,N_2,N_3;-1)$; $N_1=[0.88n]$, $N_2=[0.12n]$, $N_3=n$.

The specialization $z=-1$ corresponds to ln 2.

iii) P_n are rational integers;

iv) Q_n are rational numbers whose denominators divide $lcm\{1,\cdots,n\}$.

The asymptotics is determined according to Lemma 1.2 by roots of quartic polynomial. Numerically one has

$$\log|P_n|,\ \log|Q_n|\sim 1.5373478\cdots n$$

and

$$\log|P_n\ln 2-Q_n|\sim -1.77602924\cdot n\quad\text{as } n\to\infty.$$

Hence one has the following measure of irrationality of ln 2:

$$|q\ln 2-p|>|q|^{-3.2696549\cdots}.$$

The best exponent in the measure of diophantine approximation to ln 2 we can achieve this way requires more complicated recurrences than (3.1) corresponding to more apparent singularities:

$$|q\ln 2-p|>|q|^{-3.134400029\cdots}$$

for rational integers p, q with $|q|\geq q'$.

There exists, similarly to § 2, a three term recurrence determining the "dense" system of rational approximations to $\pi/\sqrt{3}$.

Again, there are two linearly independent solutions of the three-term recurrence: one $X_n^{(\mathrm{in})}$ is an integer solution, and another $X_n^{(\mathrm{nonin})}$ is "almost" integer solution, with denominator dividing $lcm\{1,\cdots,n\}$. We present the "integer" solution $X_n^{(\mathrm{in})}$ (the denominator in the approximation to $\pi/\sqrt{3}$) for $4|n$:

$$X_{4n}^{(\mathrm{in})}=\sum_{i_1,i_2=0,i_1+i_2<4n}^{3n}\binom{3n}{i_1}\binom{3n}{i_2}(4n-i_1-i_2)\times 3^{i_2}\cdot 2^{4n-i_2+i_1}.$$

The approximants $X_n^{(\mathrm{in})}$, $X_n^{(\mathrm{nonin})}$ possess the following properties:

I. $X_n^{(\mathrm{in})}\in Z$; denominators of $X_n^{(\mathrm{nonin})}$ are dividing $lcm\{1,\cdots,n\}$;

II. As $n\to\infty$ one has

$$\log|X_n^{(\mathrm{in})}|,\ \log|X_n^{(\mathrm{nonin})}|\longrightarrow -1.66439185\cdots n$$

and

III. $\log|(\pi/\sqrt{3})X_n^{(\mathrm{in})}+X_n^{(\mathrm{nonin})}|\to 2.2006689\cdots n$.

Hence the measure of irrationality of $\pi/\sqrt{3}$ is considerably better than it was before:

$$|q\pi/\sqrt{3}-p|>|q|^{-4.817441679\cdots}$$

as $|q|\geq q_0$.

Similarly one has better measure of irrationality of $\pi/\sqrt{3}$:

$$|q\pi/\sqrt{3}-p|>|q|^{-4.792613804\cdots}$$

for all integers p, q with $|q|\geq q_0$.

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