28. *The Nonstationary Navier-Stokes System with Some First Order Boundary Condition*

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Introduction. This paper shows that there exists a strong solution in L_p of the nonstationary Navier-Stokes system with some first order boundary condition. To prove this we study the Stokes operator with such boundary condition and use the semigroup approach in Fujita-Kato [2], [8] and Giga-Miyakawa [7].

Let D be a bounded domain in $\boldsymbol{R}^n$ with smooth boundary S. We consider the Navier-Stokes initial value problem concerning velocity $u=(u^1, \cdots, u^n)$ and pressure p:

(N) $\partial u/\partial t-\Delta u+(u, \nabla)u+\nabla p=0$, $\operatorname{div} u=0$ in $D\times(0, T)$, $u|_{t=0}=a$ in D,

where $(u, \nabla)=\sum_{j=1}^n u^j(\partial/\partial x_j)$. The boundary condition we give is

(NB) $$u\cdot\nu=0, \quad Bu=0 \quad \text{on} \quad S\times(0, T).$$

Here ν_x denotes the interior unit normal vector at $x\in S$ and $u\cdot\nu=u^1\nu^1+\cdots+u^n\nu^n$. We assume that B is a first order boundary differential operator and that $Bu\cdot\nu=0$ if $u\cdot\nu=0$.

To study this Navier-Stokes system in L_p we define the Stokes operator as follows. Let X_p $(1<p<\infty)$ denote the set of divergence free vector functions $w\in L_p(D)$ satisfying $w\cdot\nu=0$. Let P be the continuous projection from $L_p(D)$ to X_p; see [3]. Then we set $A_B=-P\Delta$ with domain $D(A_B)=\{u\in W_p^2(D);\ Bu=0\}\cap X_p$ and call A_B the *Stokes operator* with boundary condition B; here $W_p^2(D)$ denotes the Sobolev space of order two.

Concerning A_B we shall show that $-A_B$ generates an analytic semigroup in X_p if B satisfies an appropriate algebraic assumption (see the assumption (B) in § 1); the slip boundary condition is included in our case. Next we shall characterize $D((A_B+L)^\alpha)$ $(0<\alpha<1)$ for large L. We shall also study A_B^*, the dual of A_B.

Following Kato-Fujita [2], [8], we transform (N), (NB) into the evolution equation in X_p

(AN) $$du/dt+A_Bu+P(u, \nabla)u=0 \quad (t>0), \quad u(0)=a.$$

Using results on A_B, we get the existence and the uniqueness of a (local) strong solution of (AN).

Since our methods are similar to those of Giga [4]–[6] and Giga-Miyakawa [7] who studied the Dirichlet problem for (N), we do not give the detailed proof here. However, our results generalize that of

Miyakawa [9]. For different approach to (N), (NB), see Fabes-Lewis-Riviere [1], Moglievskii [10], Solonnikov [11].

In what follows we fix $1<p<\infty$ and denote by $\| \quad \|$ the norm in $L_p(D)$. We do not distinguish between the space of scalar and vector valued functions. We denote the norm in $W_p^m(S)$ by $| \quad |_m$.

1. Construction of the resolvent. First we state our assumption on the boundary operator $B=\sum_{j=1}^n b^j(\partial/\partial x_j)+d$, where b^j and d are $n\times n$ matrix-valued functions. Let $\rho_x=(\rho_{ij})$ be an $n\times n$ orthogonal matrix which maps ν_x to ${}^t(0,\cdots,0,1)$. For each $x\in S$ we denote by B_x^j $(1\leq j\leq n)$ the matrix $B_x^j=\rho(\sum_{k=1}^n \rho_{jk}b^k)\rho^{-1}$. Our assumption on B is

(B) There are some positive constants c and a such that

$$\left|\det\left[\sum_{j=1}^n B_x^j \int_{-\infty}^{\infty} \xi_j k_\lambda(\xi)\,d\xi_n\right]_{n-1}\right|>c$$

for all $x\in S$, $|\arg\lambda|\leq\pi/2$, $|\lambda|>a$, where k_λ denotes the symbol of the hydrodynamic potential K_λ; see e.g. [4]. Here $[E]_{n-1}$ denotes the $(n-1)\times(n-1)$ matrix which consists of the first $n-1$ rows and columns of an $n\times n$ matrix E.

This assumption (B) is similar to that of Fabes-Lewis-Riviere [1]. Note that the slip boundary condition is one of examples satisfying (B); see [1], [11]. In what follows, we always assume (B).

As in [5], to study the resolvent it is enough to construct $v=V_\lambda^B g$ that satisfies

$$(\lambda-\Delta)v+\nabla q=0,\quad \operatorname{div} v=0\quad \text{in } D,$$
$$v\cdot\nu=0,\qquad Bv=g\quad \text{on } S,$$

where q is some scalar function. Let Y_λ^B be a pseudo-differential operator on S of order zero. Set

$$T_\lambda^B g=BK_\lambda(\delta_S\otimes Y_\lambda^B g),\qquad T_\lambda g=\gamma K_\lambda(\delta_S\otimes Y_\lambda^B g),$$

where δ_S denotes the measure carried by S with density one and γ denotes the trace on S. Let π_τ denotes the projection such that $\pi_\tau w=w-(\nu\cdot w)\nu$ for $w\in L_p(S)$. Then the crucial step in constructing V_λ^B is

Theorem 1. *There exist a pseudo-differential operator Y_λ^B on S of order zero and a smoothing operator J such that the estimates*

$$|(T_\lambda^B-\pi_\tau-\pi_\tau J)w|_0\leq C\,|\lambda|^{-1}\,|w|_0,\qquad |\nu\cdot T_\lambda w|_1\leq C\,|\lambda|^{-1}\,|w|_0,$$
$$|Jw|_0\leq 1/2\,|w|_0\quad \text{for all } w\in L_p(S),\quad |\arg\lambda|\leq\pi/2,\quad |\lambda|>a$$

holds for some constant C.

2. Results on the Stokes operator. Now we state operator-theoretic properties of A_B (cf. [5], [6]). In general we do not know where the spectrum of A_B are, so we consider $A_B=A_B+L$ instead of A_B; here $L>0$ is so taken that the set $\{\lambda\,|\,\mathrm{Re}\,\lambda\geq 0\}$ contains no spectrum of A_B.

Theorem 2. *There are positive constants C and δ such that*

$$\|(\lambda+A_B)^{-1}f\|\leq C\,|\lambda|^{-1}\,\|f\|,\qquad f\in X_p$$

for all λ, $|\arg\lambda|\leq\pi/2+\delta$.

Domains of fractional powers A_B^α can be characterized by

Theorem 3. *$D(A_B^\alpha)$ is the complex interpolation space $[X_p, D(A_B)]_\alpha$. In particular, $D(A_B^\alpha)$ is continuously embedded in the space of Bessel potentials $H_p^{2\alpha}(D)$.*

We can prove these theorems by using Theorem 1. Since the proofs are similar to those of Theorem 1 in [5] and Theorem 2 in [6], we omit the detail.

Concerning A_B^*, the dual of A_B in X_p, we have

Proposition 1. *There is a boundary differential operator B' such that $A_B^* = A_{B'}$ as operators in $X_{p'}$, where $1/p+1/p'=1$. Moreover, B' satisfies the condition* (B).

Proof. To show this proposition it is enough to prove the same result for the Laplace operator L_B in X_p; see the proof of Theorem 3 in [3]. Taking λ in (B) sufficiently large, we see that det $[B_x^n]_{n-1}$ never vanishes. From this and Green's formula it follows that $L_B^* = L_{B'}$ for some B'. Thus we have $A_B^* = A_{B'}$. It is not difficult to show that B' satisfies (B). Q.E.D.

3. The Navier-Stokes initial value problem. By Theorem 3 and Proposition 1 we have the same estimate for the nonlinear term of (AN) as Lemma 2.2 in [7] (A should be replaced by A_B). This estimate together with Theorem 2 shows that there is a unique strong solution of (AN); see [7]. More precisely, we have

Theorem 4. *Fix γ such that $n/2p - 1/2 \leq \gamma < 1$. Assume that a is in $D(A_B^\gamma)$. Then there exists a unique local solution u of* (AN) *with the following properties.*

(i) *u is continuous from $[0, T)$ to $D(A_B^\gamma)$,*

(ii) *u is continuous from $(0, T)$ to $D(A_B^\alpha)$ and $\|A_B^\alpha u(t)\| = o(t^{\gamma-\alpha})$ as $t \to 0$ for some α, $\gamma < \alpha < 1$, for some $T > 0$.*

Moreover, u is smooth in $\bar{D} \times (0, T)$.

To prove this we put $u(t) = e^{Lt}v(t)$ in (AN). Then $v(t)$ is a solution of

$$dv/dt + A_B v + e^{Lt}P(v, \nabla)v = 0, \qquad v(0) = a.$$

Applying Theorem 2 and the estimates of nonlinear term to this equation, we get Theorem 4 in the same way as in [7].

Before concluding this paper, we consider, for example, the case that $Bu=0$, $u \cdot \nu = 0$ is the slip boundary condition. By Solonnikov-Ščadilow [12] we can take $L=0$ in Theorems 2 and 3. This implies that the solution in Theorem 4 exists globally if the initial velocity a is sufficiently small in $D(A_B^\gamma)$; see Theorem 2.6 in [7].

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