17. *Zeta Functions in Several Variables Associated with Prehomogeneous Vector Spaces.* I[*)]

Functional Equations

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1. In this note we introduce zeta functions in several variables associated with prehomogeneous vector spaces defined over the rational number field $\boldsymbol{Q}$ and discuss their functional equations and analytic continuations. Our results are generalizations to those of M. Sato and T. Shintani [4], in which they treated the zeta functions in single variable.

2. Let G be a connected linear algebraic group defined over $\boldsymbol{Q}$. Let ρ_1 and ρ_2 be $\boldsymbol{Q}$-rational representations of G on finite dimensional complex vector spaces E and F with $\boldsymbol{Q}$-structures. Put $\rho=\rho_1\oplus\rho_2$ and $V=E\oplus F$. Here we do not exclude the case where $E=\{0\}$. In the present note we always assume that (G, ρ, V) *is a prehomogeneous vector space* (briefly a p.v.) (for the definition of p.v. and other basic notions in the theory of p.v.'s, we refer to M. Sato and T. Kimura [3]). We assume further that

(A.1) *F is a $\boldsymbol{Q}$-regular subspace of* (G, ρ, V)

in the following sense.

Definition. The invariant subspace F is called a *$\boldsymbol{Q}$-regular subspace of* (G, ρ, V) if there exists a relative invariant $P(x)=P(x^{(1)}, x^{(2)})$ $(x^{(1)}\in E, x^{(2)}\in F)$ of (G, ρ, V) with coefficients in $\boldsymbol{Q}$ such that the Hessian

$$\det\left(\frac{\partial^2 P}{\partial x_i^{(2)}\partial x_j^{(2)}}(x^{(1)}, x^{(2)})\right)$$

of P with respect to the variables $x_1^{(2)}, \cdots, x_{\dim F}^{(2)}$ in F is not identically zero.

Let F^* be the vector space dual to F and ρ_2^* be the representation of G on F^* contragredient to ρ_2. Put $\rho^*=\rho_1\oplus\rho_2^*$ and $V^*=E\oplus F^*$.

Lemma 1. *The triple (G, ρ^*, V^*) is a prehomogeneous vector space and F^* is a $\boldsymbol{Q}$-regular subspace of (G, ρ^*, V^*).*

We call (G, ρ^*, V^*) *the partially dual p.v. of* (G, ρ, V) *with respect to the $\boldsymbol{Q}$-regular subspace F.*

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Let S and S^* be the singular sets of (G, ρ, V) and (G, ρ^*, V^*) respectively.

Lemma 2. (i) *The set S (resp. S^*) is a proper algebraic subset of V (resp. V^*) defined over $\boldsymbol{Q}$.*

(ii) *The number of $\boldsymbol{Q}$-irreducible components of S with codimension 1 is equal to that of S^*.*

(iii) *S is a hypersurface in V if and only if S^* is a hypersurface in V^*.*

Let $P_1, \cdots, P_n$ (resp. $Q_1, \cdots, Q_n$) be $\boldsymbol{Q}$-irreducible polynomials defining the $\boldsymbol{Q}$-irreducible components of S (resp. S^*) with codimension 1. It is known that these polynomials are relative invariants of G. Denote by $\chi_1, \cdots, \chi_n$ (resp. $\chi_1^*, \cdots, \chi_n^*$) the $\boldsymbol{Q}$-rational characters of G corresponding to $P_1, \cdots, P_n$ (resp. $Q_1, \cdots, Q_n$):

$$P_i(\rho(g)x)=\chi_i(g)P_i(x),$$
$$Q_i(\rho^*(g)x^*)=\chi_i^*(g)Q_i(x^*)$$

$(1 \leq i \leq n,\ g \in G,\ x \in V,\ x^* \in V^*)$.

Let $X_\rho(G)$ (resp. $X_{\rho^*}(G)$) be the subgroup of the group of $\boldsymbol{Q}$-rational characters of G generated by $\chi_1, \cdots, \chi_n$ (resp. $\chi_1^*, \cdots, \chi_n^*$). Then

Lemma 3. (i) *The group $X_\rho(G)$ coincides with $X_{\rho^*}(G)$.*

(ii) *The group $X_\rho(G)=X_{\rho^*}(G)$ is a free abelian group of rank n with two systems of generators $\{\chi_1, \cdots, \chi_n\}$ and $\{\chi_1^*, \cdots, \chi_n^*\}$.*

(iii) *The character* $\det \rho_2(g)^2$ *is contained in $X_\rho(G)$.*

Define an n by n unimodular matrix $U=(u_{ij})$ and an n-tuple $\lambda=(\lambda_1, \cdots, \lambda_n)$ of half-integers by the following formulas:

$$\chi_i=\prod_{j=1}^{n} \chi_j^{*u_{ij}} \quad (1 \leq i \leq n),$$
$$\det \rho_2(g)^2=\prod_{i=1}^{n} \chi_i(g)^{2\lambda_i}.$$

3. We fix a subgroup $G_{\boldsymbol{R}}^+$ of the group of real points of G containing the identity component.

Lemma 4. *The number of $\rho(G_{\boldsymbol{R}}^+)$-orbits in $V_{\boldsymbol{R}}-S_{\boldsymbol{R}}$ is equal to that of $\rho^*(G_{\boldsymbol{R}}^+)$-orbits in $V_{\boldsymbol{R}}^*-S_{\boldsymbol{R}}^*$.*

Let

$$V_{\boldsymbol{R}}-S_{\boldsymbol{R}}=V_1 \cup \cdots \cup V_\nu$$

and

$$V_{\boldsymbol{R}}^*-S_{\boldsymbol{R}}^*=V_1^* \cup \cdots \cup V_\nu^*$$

be their $G_{\boldsymbol{R}}^+$-orbital decompositions.

Denote by $\mathcal{S}(V_{\boldsymbol{R}})$ and $\mathcal{S}(V_{\boldsymbol{R}}^*)$ the spaces of rapidly decreasing functions on $V_{\boldsymbol{R}}$ and $V_{\boldsymbol{R}}^*$ respectively. Let $dx^{(1)}$, $dx^{(2)}$ and $dx^{*(2)}$ be Euclidean measures on $E_{\boldsymbol{R}}$, $F_{\boldsymbol{R}}$ and $F_{\boldsymbol{R}}^*$ respectively. Put

$$dx=dx^{(1)}dx^{(2)} \quad (x=(x^{(1)}, x^{(2)}) \in V_{\boldsymbol{R}})$$

and

$$dx^*=dx^{(1)}dx^{*(2)} \quad (x^*=(x^{(1)}, x^{*(2)}) \in V_{\boldsymbol{R}}^*).$$

Set

$$\Phi_j(f\,;s)=\int_{V_j}\prod_{i=1}^{n}|P_i(x)|^{s_i}f(x)dx$$

and

$$\Phi_j^*(f^*\,;s)=\int_{V_j^*}\prod_{i=1}^{n}|Q_i(x^*)|^{s_i}f^*(x^*)dx^*$$

$$(1\leqq j\leqq \nu,\ f\in\mathcal{S}(V_{\boldsymbol{R}}),\ f^*\in\mathcal{S}(V_{\boldsymbol{R}}^*),\ s=(s_1,\cdots,s_n)\in\boldsymbol{C}^n).$$

The integrals Φ_j and Φ_j^* converge absolutely for $\mathrm{Re}\,s_1,\cdots,\mathrm{Re}\,s_n>0$ and have analytic continuations meromorphic functions of s in $\boldsymbol{C}^n$ (cf. I.N. Bernstein and S.I. Gelfand [1]).

Define a partial Fourier transform $\hat{f}^*$ of $f^*\in\mathcal{S}(V_{\boldsymbol{R}}^*)$ with respect to F^* by setting

$$\hat{f}^*(x)=\hat{f}^*(x^{(1)},x^{(2)})=\int_{F_{\boldsymbol{R}}^*}f^*(x^{(1)},x^{*(2)})\exp(2\pi i\langle x^{(2)},x^{*(2)}\rangle)dx^{*(2)}.$$

Theorem 1. *In addition to* (A.1), *suppose that*

(A.2) *the singular set S of (G,ρ,V) is a hypersurface in V.*

Then the functions $\Phi_1,\cdots,\Phi_\nu$ and $\Phi_1^*,\cdots,\Phi_\nu^*$ satisfy the following functional equations:

$$\Phi_i(\hat{f}^*\,;s)=\left(\prod_{i=1}^{n}c_i^{-s_i}\right)(2\pi i)^{d^*(s)}\gamma(s)\sum_{j=1}^{\nu}a_{ij}(s)\Phi_j^*(f^*\,;(s+\lambda)U)$$
$$(f^*\in\mathcal{S}(V_{\boldsymbol{R}}^*)).$$

Here $c_1,\cdots,c_n$ are non-zero complex numbers,

$d^*(s)=d_1s_1+\cdots+d_ns_n$ *with*

$d_i=$*the degree of $Q_i(x^{(1)},x^{*(2)})$ with respect to $x^{*(2)}$,*

$\gamma(s)$ is a Gamma factor of the form

$$\gamma(s)=\prod_k\Gamma(L_k(s))^{\sigma_k}\quad(\sigma_k=1\text{ or }-1)$$

for some inhomogeneous linear forms $L_k(s)$ in s, and $a_{ij}(s)$ are polynomial functions in $\exp(\pm\pi is_1),\cdots,\exp(\pm\pi is_n)$.

The theorem is a gereralization of Theorem 4 in [2], Theorem 1 in [4] and Theorem 1.1 in [5]. By suitably modifying the argument in [2] and [5], we are able to show the theorem.

4. Put

$$\Gamma=\{g\in G_{\boldsymbol{Z}}\cap G_{\boldsymbol{R}}^+\,;\ \chi(g)=1\quad\text{for all}\quad\chi\in X_\rho(G)=X_{\rho^*}(G)\}.$$

Let M and N be Γ-invariant lattices in $E_{\boldsymbol{Q}}$ and $F_{\boldsymbol{Q}}$ respectively. Denote by N^* the lattice in $F_{\boldsymbol{Q}}^*$ dual to N. Put $L=M\oplus N$ and $L^*=M\oplus N^*$. The lattice L (resp. L^*) is a Γ-invariant lattice in $V_{\boldsymbol{Q}}$ (resp. $V_{\boldsymbol{Q}}^*$). Let dg be a right invariant measure on $G_{\boldsymbol{R}}^+$. Define a character Δ of $G_{\boldsymbol{R}}^+$ by the formula

$$d(hg)=\Delta(h)dg.$$

We assume that

(A.3) *the integrals*

$$Z(f,L\,;s)=\int_{G_{\boldsymbol{R}}^+/\Gamma}\prod_{i=1}^{n}|\chi_i(g)|^{s_i}\sum_{x\in L-S}f(\rho(g)x)dg\quad(f\in\mathcal{S}(V_{\boldsymbol{R}}))$$

and

$$Z^*(f^*, L^*; s)=\int_{G_R^+/\Gamma} \prod_{i=1}^{n} |\chi_i^*(g)|^{s_i} \sum_{x^*\in L^*-S^*} f^*(\rho^*(g)x^*)dg$$

$$(f^* \in \mathcal{S}(V_R^*))$$

are convergent absolutely when $\mathrm{Re}\, s_1, \cdots, \mathrm{Re}\, s_n$ *are sufficiently large.*

For an $x \in V_Q$ (resp. $x^* \in V_Q^*$), let G_x (resp. G_{x^*}) be the isotropy subgroup of G at x (resp. x^*) and put

$$G_x^+=G_x\cap G_R^+,\quad \Gamma_x=G_x\cap\Gamma,\quad G_{x^*}^+=G_{x^*}\cap G_R^+,\quad \Gamma_{x^*}=G_{x^*}\cap\Gamma.$$

By the assumption (A.3), we get the next lemma.

Lemma 5. (i) *For any* $x \in V_Q-S_Q$ *(resp.* $x^* \in V_Q^*-S_Q^*$*), the group* G_x^+ *(resp.* $G_{x^*}^+$*) is a unimodular Lie group and the volume of* G_x^+/Γ_x *(resp.* $G_{x^*}^+/\Gamma_{x^*}$*) with respect to a Haar measure is finite.*

(ii) *There exist* $\delta=(\delta_1, \cdots, \delta_n)$ *and* $\delta^*=(\delta_1^*, \cdots, \delta_n^*)$ *in* $\boldsymbol{Q}^n$ *such that*

$$|\det \rho(g)|\Delta(g)^{-1}=|\chi_1(g)|^{\delta_1}\cdots|\chi_n(g)|^{\delta_n}$$

and

$$|\det \rho^*(g)|\Delta(g)^{-1}=|\chi_1^*(g)|^{\delta_1^*}\cdots|\chi_n^*(g)|^{\delta_n^*}$$

for all $g \in G_R^+$.

It is easy to see that $\delta^*=(\delta-2\lambda)U$.

For any $x \in V_Q-S_Q$ (resp. $x^* \in V_Q^*-S_Q^*$), normalize a Haar measure $d\mu_x$ (resp. $d\mu_{x^*}$) on G_x^+ (resp. $G_{x^*}^+$) by the formula

$$\int_{G_R^+} F(g)dg=\int_{G_R^+/G_x^+} \prod_{i=1}^{n} |P_i(\rho(g)x)|^{-\delta_i}d(\rho(g)x) \int_{G_x^+} F(gh)d\mu_x(h)$$

$$\left(\text{resp.} \int_{G_R^+} F(g)dg=\int_{G_R^+/G_{x^*}^+} \prod_{i=1}^{n} |Q_i(\rho^*(g)x^*)|^{-\delta_i^*}d(\rho^*(g)x^*)\int_{G_{x^*}^+} F(gh)d\mu_{x^*}(h)\right)$$

$(F \in L^1(G_R^+, dg))$.

Set $L_i=L\cap V_i$ and $L_i^*=L^*\cap V_i^*$ $(1\leqq i\leqq \nu)$. Denote by $\Gamma\backslash L_i$ (resp. $\Gamma\backslash L_i^*$) the set of all Γ-orbits in L_i (resp. L_i^*). Also set

$$\xi_j(L; s)=\sum_{x\in\Gamma\backslash L_j} \mu(x) \prod_{i=1}^{n} |P_i(x)|^{-s_i}, \quad \mu(x)=\int_{G_x^+/\Gamma_x} d\mu_x$$

and

$$\xi_j^*(L^*; s)=\sum_{x^*\in\Gamma\backslash L_j^*} \mu(x^*) \prod_{i=1}^{n} |Q_i(x^*)|^{-s_i}, \quad \mu(x^*)=\int_{G_{x^*}^+/\Gamma_{x^*}} d\mu_{x^*}$$

$(1\leqq j\leqq \nu,\ s \in \boldsymbol{C}^n)$.

Lemma 6. *Let* B *(resp.* B^**) be the domain of absolute convergence of* $Z(f, L; s)$ *(resp.* $Z^*(f^*, L^*; s)$*). Then the Dirichlet series* $\xi_1(L; s), \cdots, \xi_\nu(L; s)$ *(resp.* $\xi_1^*(L^*; s), \cdots, \xi_\nu^*(L^*; s)$*) are convergent absolutely for* $s \in B$ *(resp.* $s \in B^*$*). Moreover the following equalities hold:*

$$Z(f, L; s)=\sum_{i=1}^{\nu} \xi_i(L; s)\Phi_i(f; s-\delta) \quad (s \in B,\ f \in \mathcal{S}(V_R)),$$

$$Z^*(f^*, L^*; s)=\sum_{i=1}^{\nu} \xi_i^*(L^*; s)\Phi_i^*(f^*; s-\delta^*) \quad (s \in B^*,\ f^* \in \mathcal{S}(V_R^*)).$$

Definition. The series $\xi_1(L;s), \cdots, \xi_\nu(L;s)$ (resp. $\xi_1^*(L^*;s), \cdots, \xi_\nu^*(L^*;s)$) are called *the zeta functions associated with* (G, ρ, V) *and* L (resp. (G, ρ^*, V^*) *and* L^*).

Let D (resp. D^*) be the convex hull of $(B^*U^{-1}+\lambda)\cup B$ (resp. $(B-\lambda)U\cup B^*$). Then Theorem 1 and Lemma 6 yield the following theorem.

Theorem 2. *If the conditions* (A.1), (A.2) *and* (A.3) *hold, then*

(i) *the series* $\xi_1(L;s), \cdots, \xi_\nu(L;s)$ (*resp.* $\xi_1^*(L^*;s), \cdots, \xi_\nu^*(L^*;s)$) *have analytic continuations to meromorphic functions of* s *in* D (*resp.* D^*).

(ii) *The following functional equations hold for* $s\in D$:

$$\xi_i^*(L^*;(s-\lambda)U)=v(N)\left(\prod_{i=1}^{n} c_i^{\delta_i-s_i}\right)(-2\pi i)^{d^*(s-\delta)}\gamma(s-\delta)$$
$$\sum_{j=1}^{\nu} a_{ji}(s-\delta)\xi_j(L;s) \quad (1\leqq i\leqq \nu)$$

where $v(N)=\int_{F_{\boldsymbol{R}}/N} dx^{(2)}$.

By Theorem 2 and Proposition 24, Remark 26 of [3, § 4], we obtain the next theorem.

Theorem 3. *Let* (G, ρ, V) *be a p.v. satisfying the conditions* (A.1) *and* (A.3) *for* $E=\{0\}$ *and* $F=V$. *Assume further that* G *is a reductive algebraic group. Then the zeta functions* $\xi_1(L;s), \cdots, \xi_\nu(L;s)$ *associated with* (G, ρ, V) *and a* Γ*-invariant lattice* L *in* V_Q *have analytic continuations to meromorphic functions of* s *in* $\boldsymbol{C}^n$.

Remarks. (1) Theorems 2 and 3 were proved in [4] under the assumptions that $E=\{0\}$, G is reductive and S is an absolutely irreducible hypersurface (hence $n=1$).

(2) It frequently occurs that a given p.v. has several $\boldsymbol{Q}$-regular subspaces. Then the associated zeta functions satisfy a number of functional equations.

(3) The result of T. Suzuki [6] can be regarded as a concrete example of our theory. The full exposition of this paper with some other examples will appear elsewhere.

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