# 14. *A Summation Method for Linearly Ordered Series*

By Kanesiroo ISEKI
Department of Mathematics, Ochanomizu University, Tokyo

(Communicated by Shokichi IYANAGA, M. J. A., Jan. 12, 1981)

By a *linearly ordered sequence* with real-valued terms we understand an arbitrary real-valued function $f$ defined on an indexing set $\Omega$, where we suppose $\Omega$ to be nonvoid, at most countable, and furnished with a linear order. We are concerned with a summation method by which, under certain conditions, the terms $f(\omega)$ of a linearly ordered sequence can be summed in the given order of the indices $\omega$. When we deal with summation, we shall speak of a linearly ordered *series* instead of *sequence*, in accordance with the usual wording.

Given $\Omega$ as above, there always exists a linear compact set $Q$ of measure zero, containing at least two points and such that $\Omega$ can be mapped biuniquely and order-isomorphically onto the aggregate $\mathfrak{M}$ of the open intervals contiguous to $Q$, where we suppose $\mathfrak{M}$ to be linearly ordered by its natural ordering. This set $Q$ and the relevant order-isomorphism will be kept fixed in the sequel.

We shall say that a linearly ordered series $\sum_\omega f(\omega)$ defined on $\Omega$ is *Luzin convergent*, if there exists at least one real-valued function $F(x)$ $(-\infty<x<+\infty)$ subject to the following three conditions and if, further, two such functions $F$ necessarily differ, over the set $Q$, only by an additive constant.

(i) The function $F(x)$ is continuous on the set $Q$;

(ii) the image $F[Q]$ of $Q$ under $F$ is of measure zero;

(iii) we have $f(\omega)=F(v)-F(u)$, for every $\omega\in\Omega$, where $(u, v)$ is that open interval (contiguous to $Q$) which corresponds to the index $\omega$ under the order-isomorphism considered above.

When this is the case, we define the *Luzin sum* of our series by the formula $\sum_\omega f(\omega)=F(b)-F(a)$, where $[a, b]$ denotes the minimal closed interval containing the set $Q$. Clearly, this definition is independent of the choice of the function $F$. The epithet "Luzin" indicates the importance of the condition (N) of Luzin in our theory.

It can be shown that the above notion of Luzin convergence, as well as the value of the Luzin sum, depends neither on the choice of the set $Q$ nor on that of the relevant order-isomorphism.

We shall call a linearly ordered series $\sum_\omega f(\omega)$ defined on an indexing set $\Omega$ to be *Denjoy convergent*, if there exists a real-valued function $F(x)$ $(-\infty<x<+\infty)$ which is generalized absolutely continuous on the set $Q$ and for which the relation $f(\omega)=F(v)-F(u)$ holds for every $\omega \in \Omega$, where $(u, v)$ means the open interval corresponding to $\omega$ under the aforesaid order-isomorphism. This definition does not depend on the choice of the set $Q$ or of the relevant order-isomorphism. The epithet "Denjoy" indicates that the notion of Denjoy convergence is closely related to the Denjoy integration.

*Every linearly ordered series which is Denjoy convergent, is necessarily Luzin convergent.* This is the main result of our theory. At present, however, we are unable to decide whether or not *there exist linearly ordered series which are Luzin convergent without being Denjoy convergent.*

A detailed account of the contents of this note will appear in the forthcoming number of the Natural Science Report of the Ochanomizu University.

## Reference

S. Saks: Theory of the Integral. Warszawa-Lwów (1937).