117. An Estimate of the Roots of b-Functions by Newton Polyhedra

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(Communicated by Kunihiko KODAIRA, M. J. A., Dec. 12, 1981)

Introduction. In this note we give an estimate of the roots of b-functions of certain isolated singularities (Theorem 4.4).

The theory of b-functions and the proof given here are based on Yano [5]. In the real analytic case, the same estimate is given in Varchenko [4].

The author is grateful to Dr. Tamaki Yano for many valuable advices.

§ 1. Let $\mathcal{O}$ be the set of germs of holomorphic functions at the origin O of C^{n+1}, $\mathcal{D}=\mathcal{O}[\partial/\partial x_0, \cdots, \partial/\partial x_n]$, $B_{pt}=D\delta$ where δ is the δ-function.

For any $f \in \mathcal{O}$, there exist $P(s) \in \mathcal{D}[s]$, $b(s) \in C[s]$ such that $P(s)f^{s+1} = b(s)f^s$ (Bernstein [1], Björk [2]). These $b(s)$ form an ideal and the generator of the ideal is called the *b-function* of f and denoted by $b_f(s)$. If $f(0)=0$, $b_f(s)$ is divided by $s+1$ and we put $\tilde{b}_f(s)=b_f(s)/(s+1)$. $\mathcal{J}_f(s) =\{P(s) \in \mathcal{D}[s] : P(s)f^s=0\}$.

Let $\Gamma_+(f)$ be the Newton polyhedron of f and $\{\gamma_1, \cdots, \gamma_m\}$ the set of all the n-dimensional faces of $\Gamma_+(f)$ not contained in $\{x : \prod_{i=0}^n x_i=0\}$, $\gamma_k=\{(x_0, \cdots, x_n) : \sum d_{k,i}x_i=1\}$. Then $d_k(x_i)=d_{k,i}$ defines a degree on $\mathcal{O}$, and we put $X_k=\sum d_{k,i}x_i\partial/\partial x_i$.

§ 2. From now on we assume that $f \in \mathcal{O}\,(f(0)=0)$ has an isolated singularity and is nondegenerate with respect to $\Gamma_+(f)$.

2.1. Theorem (Kashiwara-Yano). *α is a root of $\tilde{b}_f(s)$ if and only if there exists a nonzero element Δ of B_{pt} satisfying the following two conditions*:

(2.1.1) $\quad f(x)\Delta=0$ *and* $\partial f/\partial x_i\Delta=0, \quad i=0, \cdots, n,$

(2.1.2) $\quad$ *for any* $P(s) \in \mathcal{J}_f(s), \quad P(\alpha)\Delta=0.$

2.2. Theorem (Teissier [3]). *For any ideal I of $\mathcal{O}$, there exists $\nu_0 \in N$ such that, for any $\nu \in N$, $\overline{I^{\nu+\nu_0}}=I^\nu \cdot \overline{I^{\nu_0}}$, where $\bar{I}$ denotes the integral closure of I.*

2.3. Proposition. *Let $I=(x_0\partial f/\partial x_0, \cdots, x_n\partial f/\partial x_n)\mathcal{O}$. For any $\nu \in N$ and $g \in \mathcal{O}$, $g \in \overline{I^\nu}$ if and only if $d_k(g)\geqq\nu$, $k=1, \cdots, m$.*

§ 3. Construction of an operator $P(s) \in \mathcal{J}_f(s)$. An element of $\mathcal{D}[s]f^s$ is uniquely expressed as a finite sum $\sum_i a_i(x)f[i]$, $a_i \in \mathcal{O}$, $f[i]$

$=s(s-1)\cdots(s-i+1)f^{s-i}$, $f[0]=f^s$.

3.1. Definition. $d_k(\sum a_i(x)f[i])=\inf\{d_k(a_i)-i\}$, $k=1,\cdots,m$.

3.2. Proposition. *If* $d_k(\sum a_i f[i])=d<\infty$, *then we have the inequality* $d_k((s+d-X_k)(\sum a_i f[i]))>d$.

Proof. By 3.1, $d_k(a_i)\geqq d+i$, and we let a_i^* be the d_k-homogeneous part of a_i of degree $d+i$.

(3.2.1) $(s-X_k)(\sum a_i f[i])=\sum\{a_i(f-X_k f)f[i+1]+(i-X_k)a_i f[i]\}$.

Since $d_k(f-X_k f)>1$, the d_k-homogeneous part of (3.2.1) of degree d is $\sum(i-X_k)a_i^* f[i]=-d\sum a_i^* f[i]$. Thus the d_k-homogeneous part of $(s+d-X_k)(\sum a_i f[i])$ of degree d is zero.

3.3. Let $I=(x_0\partial f/\partial x_0,\cdots,x_n\partial f/\partial x_n)\mathcal{O}$ and let $\nu_0\in\boldsymbol{N}$ have the property of Theorem 2.2.

We put $P_0(s)=1$, and for $k\geqq 1$, we define $P_k(s)$ inductively:

(3.3.1) $$P_k(s)=(s+d_{k'}(P_{k-1}f^s)-X_{k'})P_{k-1},$$

where k' is such an integer that $1\leqq k'\leqq m$ and $k'\equiv k \bmod m$.

Choose $N\in\boldsymbol{N}$ so that

(3.3.2) $$d_k(P_N f^s)\geqq\nu_0,\qquad k=1,\cdots,m.$$

Let

(3.3.3) $$P_N f^s=\sum_{i=1}^N a_i f[i].$$

Then (3.3.2) implies

(3.3.4) $$d_k(a_i)\geqq\nu_0+i,\quad i=1,\cdots,N,\quad k=1,\cdots,m,$$

or equivalently

(3.3.4)′ $$a_i\in I^i\cdot\overline{I^{\nu_0}},\qquad i=1,\cdots,N.$$

Let

(3.3.5) $$a_N=\sum_{|J|=N}u_J(x)(x_i\partial f/\partial x_i)^J,\qquad u_J\in\overline{I^{\nu_0}}.$$

Put

(3.3.6) $$P'_N=P_N-\sum_{|J|=N}u_J(x)(x_i\partial/\partial x_i)^J.$$

Then

(3.3.7) $$P'_N f^s=\sum_{i=1}^{N-1}a'_i(x)f[i],\qquad a'_i\in I^i\cdot\overline{I^{\nu_0}}.$$

Continuing the same reduction N times, we get an operator $P(s)\in\mathcal{J}_f(s)$ of the form:

(3.3.8) $$P(s)=P_N(s)-\sum_{|J|\leqq N}u_J(x)(x_i\partial/\partial x_i)^J,\qquad u_J\in\overline{I^{\nu_0}}.$$

§ 4. An estimation of roots of $\tilde{b}_f(s)$.

4.1. Lemma. *Let* $P\in\mathcal{D}$, $\Delta\in B_{pt}$, $P\Delta=0$, *and let* d *be a degree on* $\mathcal{D}, B_{pt}$ $(d(\delta)=0)$. *Let* P^*,Δ^* *be, respectively, the homogeneous parts of* P,Δ *of the lowest degrees. Then we have* $P^*\Delta^*=0$.

4.2. Lemma. *Assume that* $\Delta\in B_{pt}$ *is homogeneous with respect to* d_k, *then* $X_k\Delta=(d_k(\Delta)-\sum_i d_k(x_i))\Delta$.

4.3. Let α be a root of $\tilde{b}_f(s)$ and let $P(s)\in\mathcal{J}_f(s)$ be that of (3.3.8). By Theorem 2.1, there exists $\Delta\in B_{pt}$ with the properties (2.1.1), (2.1.2). In particular $P(\alpha)\Delta=0$.

Recall that P_N is the homogeneous part of P of degree 0 with respect to all d_k and $d_k(P-P_N)\geqq\nu_0$.

Let Δ_k be the d_k-homogeneous part of Δ_{k-1} of the lowest degree, $\Delta_0=\Delta$. Then by 4.1, $P_N\Delta_k=0$ for $k\geqq 1$.

Since Δ_m is homogeneous with respect to all d_k, we have, by 4.2,

(4.3.1) $\quad P_N(\alpha)\Delta_m=\prod_{k=1}^{N}(\alpha+d'_k(P_{k-1}f^s)-d'_k(\Delta_m)+\sum_i d'_k(x_i))\Delta_m.$

Hence we obtain

(4.3.2) $$-\alpha\geqq\sum_i d_k(x_i),\qquad k=1,\cdots,m.$$

4.4. Theorem. *Assume that* $f\in\mathcal{O}$ *has an isolated singularity and is nondegenerate with respect to* $\Gamma_+(f)$. *Put* $t_0=\inf\{t:(t,\cdots,t)\in\Gamma_+(f)\}$. *Then, for any root* α *of* $\tilde{b}_f(s)$, *we have* $-\alpha\geqq t_0^{-1}$.

Proof. t_0^{-1} is equal to $\inf_k\{\sum_i d_k(x_i)\}$ and (4.3.2) proves the theorem.

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