112. An Explicit Solution of a Certain Schwinger-Dyson Equation

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1. Statement of the result. In this note, we give an explicit solution of the following equation.

$$(1.1)_N \quad \begin{cases} \left(\dfrac{d^2}{dt^2}+\omega^2\right)\dfrac{\delta Z(p,u)}{\delta p_\alpha(t)}=-ip_\alpha(t)Z(p,u)+\lambda\displaystyle\int_{R^3}\rho(x)\frac{\delta Z(p,u)}{\delta u(x,t)}dx \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \alpha=1,2,\cdots,N \\ \square\dfrac{\delta Z(p,u)}{\delta u(x,t)}=-iu(x,t)Z(p,u)+\lambda\rho(x)\displaystyle\sum_{\alpha=1}^{N}\frac{\delta Z(p,u)}{\delta p_\alpha(t)} \end{cases}$$

with subsidary conditions given by

$$(1.2)_N \quad \begin{cases} Z(0,0)=1, \\ \lim\limits_{\lambda\to 0}\dfrac{\delta^2 Z(0,0)}{\delta p_\alpha(t_1)\delta p_\alpha(t_2)}=\varDelta_F^H(t_1-t_2) \qquad \text{for any } \alpha=1,2,\cdots,N, \\ \lim\limits_{\lambda\to 0}\dfrac{\delta^2 Z(0,0)}{\delta u(x_1,t_1)\delta u(x_2,t_2)}=\varDelta_F^B(x_1-x_2,t_1-t_2). \end{cases}$$

Here $Z(p,u)$ is an unknown complex valued functional of (p,u), $p(t)=(p_1(t),p_2(t),\cdots,p_N(t))$ and $u(x,t)$ are real valued functions on $\boldsymbol{R}$ and $\boldsymbol{R}^3\times\boldsymbol{R}$, respectively. $\rho(x)$ is a given radially symmetric real valued function in $C_0^\infty(\boldsymbol{R}^3)$, ω and λ are positive constants called the (renormalized) spring constant and the coupling constant respectively, $\square$ stands for $\square v=v_{tt}-\sum_{j=1}^3 v_{x_jx_j}$ and the symbols $\delta/\delta p(t)$ and $\delta/\delta u(x,t)$ are functional (or Volterra) derivatives. $\varDelta_F^H(t)$ and $\varDelta_F^B(x,t)$ are distributions, called Feynman propagators, given by

$$(1.3) \qquad \varDelta_F^H(t)=\frac{1}{2\omega}(\theta(t)e^{-i\omega t}+\theta(-t)e^{i\omega t})$$

and

$$(1.4) \qquad \varDelta_F^B(x,t)=\frac{1}{2(2\pi)^3}\int_{R^3}\frac{1}{|\xi|}(\theta(t)e^{-i|\xi|t+ix\cdot\xi}+\theta(-t)e^{i|\xi|t-ix\cdot\xi})d\xi$$

where $\theta(t)$ is the Heaviside function. We denote the operators having as its kernel $\varDelta_F^H(t)$ and $\varDelta_F^B(x,t)$ by $(d^2/dt^2+\omega^2)_F^{-1}$ and $\square_F^{-1}$ respectively.

This equation stems from 'quantizing' the following Lagrangean by the method of Schwinger-Dyson.

$$(1.5)_N \qquad \begin{aligned} L(q,v)=&\sum_{\alpha=1}^{N}\int_R\left(\frac{1}{2}\dot q_\alpha^2(t)-\frac{1}{2}\omega_0^2 q_\alpha^2(t)\right)dt \\ &+\int_R\int_{R^3}\left(\frac{1}{2}v_t^2(x,t)-\frac{1}{2}|\nabla v(x,t)|^2\right)dxdt \end{aligned}$$

$$-\lambda\int_{R}\int_{R^3}\rho(x)v(x,t)\sum_{\alpha=1}^{N}q_\alpha(t)dxdt$$

where

$$q=(q_1,q_2,\cdots,q_N),\qquad |\nabla v(x,t)|^2=\sum_{j=1}^{3}|v_{x_jx_j}(x,t)|^2$$

and $\dot{q}_\alpha(t)=(d/dt)q_\alpha(t)$. The equation $(1.1)_N$ is obtained from formal calculation of the 'imaginary' expression of $Z(p,u)$ given by

$$(1.6)\quad Z(p,u)=\frac{\int \exp\frac{i}{h}L(q,v)\cdot\exp-\frac{i}{h}(\langle q,v\rangle+\langle v,u\rangle)d_F(q)d_F(v)}{\int\exp\frac{i}{h}L(q,v)d_F(q)d_F(u)}$$

where

$$\langle q,p\rangle=\int_{R}\sum_{\alpha=1}^{N}q_\alpha(t)p_\alpha(t)dt,\qquad \langle v,u\rangle=\int_{R^3}v(x,t)u(x,t)dxdt$$

and $d_F(q)=\prod_{\alpha=1}^{N}d_F(q_\alpha)$ and $d_F(v)$ are 'Feynman measures'.

In order to state our theorem, we need the following

Proposition. *If we define ω^2 as*

$$(1.7)_N\qquad \omega^2=\omega_0^2+\lambda^2N\frac{1}{(2\pi)^3}\int_{R^3}\frac{|\hat{\rho}(\xi)|^2}{|\xi|^2}d\xi,\qquad \hat{\rho}(\xi)=\int_{R^3}\rho(x)e^{-ix\cdot\xi}dx,$$

then the operator $A_{\lambda,N}$ defined below is invertible as an operator from $H^1(\boldsymbol{R})$ to $H^{-1}(\boldsymbol{R})$.

$$(1.8)_N\qquad \begin{aligned}(A_{\lambda,N}v)(t)=&\left(\frac{d^2}{dt^2}+\omega^2\right)v(t)\\ &-\lambda^2N\int_{R^3}\int_{R}\int_{R^3}\rho(x)\Delta_F^B(x-y,t-s)\rho(y)v(s)dydsdx\end{aligned}$$

for $v\in H^1(\boldsymbol{R})$.

Theorem. *Define ω^2 as above, then the functional $Z(p,u)$ given below is well defined on $(H^{-1}(\boldsymbol{R}))^N\times\mathcal{S}(\boldsymbol{R}^3\times\boldsymbol{R})$ and satisfies $(1.1)_N$ with $(1.2)_N$.*

$$(1.9)_N\qquad \begin{aligned}Z(p,u)=\exp\Bigg[&-\frac{i}{2}\sum_{\alpha=1}^{N}\left(A_{\lambda,N}^{-1}p_\alpha,\ A_{\lambda,N-1}\left(\frac{d^2}{dt^2}+\omega^2\right)_F^{-1}p_\alpha\right)\\ &-i\lambda^2\sum_{\alpha<\beta}\left(A_{\lambda,N}^{-1}p_\alpha,\left(\frac{d^2}{dt^2}+\omega^2\right)_F^{-1}(\rho,\Box_F^{-1}(\rho p_\beta))\right)-\frac{i}{2}(\Box_F^{-1}u,u)\\ &-\frac{i}{2}N\lambda^2(A_{\lambda,N}^{-1}(\rho,\Box_F^{-1}u),(\rho,\Box_F^{-1}u))\\ &+i\lambda\left(A_{\lambda,N}^{-1}\left(\sum_{\alpha=1}^{N}p_\alpha\right),(\rho,\Box_F^{-1}u)\right)\Bigg],\end{aligned}$$

where the meaning of brackets are given in the proof.

When $N=1$, there are several articles treating the second quantized problem corresponding to $(1.5)_1$. (For example, Arai [1], [2], Schwahb-Thirring [6], they use the Fock space representation. It seems rather rare to treat the equation $(1.1)_1$ directly even in physical articles.) Physically we may say that the second quantized problem of $(1.5)_1$

means a system of one 1-dimensional non-relativistic oscillator interacting with the quantized massless neutral scalar field in $\boldsymbol{R}^3$ at $x=0$ if we may take $\rho(x)=\delta(x)$.

Our point is to solve the equation $(1.1)_N$ directly without managing (1.6) and this point of view will be helpful towards the proposal by Gelfand [4].

Detailed proof will be appeared elsewhere.

2. Sketch of the proof. (For simplicity, we treat only the case $N=1$.) As the independent variables $p(\cdot)$ and $u(\cdot,\cdot)$ are defined on different spaces we must separate x and t in order to obtain equal footing independent variables. To do so, we approximate $\boldsymbol{R}^3$ by a big box Ω containing the support of $\rho(x)$ and using the eigenfunctions $\{w_j(x)\}$ of $-\Delta$, we express u as $u(x,t)=\sum_{j=1}^{\infty} u_j(t)w_j(x)$ and identify u with $\{u_j\}$. Here, $\{w_j\}$ are defined by

$$(2.1)\qquad \begin{cases} -\Delta w_j(x)=\mu_j^2 w_j(x) & \text{in } \Omega, \\ w_j(x)|_{\partial\Omega}=0. \end{cases}$$

By simple calculation, we deduce the following from $(1.1)_1$.

$$(2.2)\qquad \begin{cases} A\,\dfrac{\delta Z_\Omega(p,u)}{\delta p(t)}=-ip(t)Z_\Omega(p,u)-\lambda\sum_{j=1}^{\infty}\rho_j\dfrac{\delta Z_\Omega(p,u)}{\delta u_j(t)} \\ A_j\dfrac{\delta Z_\Omega(p,u)}{\delta u_j(t)}=-iu_j(t)Z_\Omega(p,u)-\lambda\rho_j\dfrac{\delta Z_\Omega(p,u)}{\delta p(t)} \end{cases}$$

where

$$A=\frac{d^2}{dt^2}+\omega^2,\quad A_j=\frac{d^2}{dt^2}+\mu_j^2\quad\text{and}\quad \rho_j=\int_\Omega \rho(x)w_j(x)dx.$$

Lemma. *The solution of* (2.2) *with subsidary conditions* (2.4) *given below is*

$$(2.3)\qquad \begin{aligned} Z_\Omega(p,u)=\exp\Big[&-\frac{i}{2}(\tilde{A}_\lambda^{-1}p,p)-\frac{i}{2}(\Box_{F,\Omega}^{-1}u,u) \\ &-\frac{i}{2}\lambda^2(\tilde{A}_\lambda^{-1}(\rho,\Box_{F,\Omega}^{-1}u),(\rho,\Box_{F,\Omega}^{-1}u))+i\lambda((\rho,\Box_{F,\Omega}^{-1}u),\tilde{A}_\lambda^{-1}p)\Big] \end{aligned}$$

$$(2.4)\qquad \begin{cases} Z_\Omega(0,0)=1 \\ \lim\limits_{\lambda\to 0}\dfrac{\delta^2 Z_\Omega(0,0)}{\delta p(t)\delta p(s)}=\Delta_F^B(t-s),\qquad \lim\limits_{\lambda\to 0}\dfrac{\delta Z_\Omega(0,0)}{\delta u_j(t)\delta u_j(s)}=A_{j,F}^{-1}(t-s). \end{cases}$$

Where the kernels of the operators $\Box_{F,\Omega}^{-1}$ *and* $A_{j,F}^{-1}$ *are given by*

$$(2.5)\qquad \Box_{F,\Omega}^{-1}(x,y,t)=\sum_{j=1}^{\infty}\frac{w_j(x)w_j(y)}{2\mu_j}(\theta(t)e^{-i\mu_j t}+\theta(-t)e^{i\mu_j t})$$

and

$$(2.6)\qquad A_{j,F}^{-1}(t)=\frac{1}{2\mu_j}(\theta(t)e^{-i\mu_j t}+\theta(-t)e^{i\mu_j t})$$

and the operator $\tilde{A}_\lambda$ *is defined by*

$$(2.7)\qquad (\tilde{A}_\lambda q)(t)=(Aq)(t)-\lambda^2\int_\Omega \rho(x)(\Box_{F,\Omega}^{-1}(\rho q))(x,t)dx$$

$$=(Aq)(t)-\lambda^2 \sum_{j=1}^{\infty} \rho_j^2 (A_{j,F}^{-1} q)(t).$$

After making Ω to R^3 in (2.3), we have the desired result $(1.9)_1$. Above procedure explains the construction of $(1.9)_1$ rather formally.

Now, we give the meaning to the brackets in $(1.9)_1$. Applying the Fourier transform in t in $(1.8)_1$ and using Plancherel formula in space variables, we have

$$(2.8)_1 \quad D_+(\tau^2)\hat{q}(\tau)=\left[(-\tau^2+\omega^2)+\lambda^2\frac{1}{(2\pi)^3}\int_{R^3}\frac{|\hat{\rho}(\xi)|^2}{\tau^2-|\xi|^2+i0}d\xi\right]\hat{q}(\tau)=\hat{p}(\tau)$$

where

$$\hat{q}(\tau)=\int_R e^{i\tau t}q(t)dt, \qquad \hat{w}(\xi)=\int_{R^3} e^{-ix\cdot\xi}w(x)dx.$$

If we define ω^2 as in Proposition, $D_+(\tau^2)$ never vanishes for $\tau \in R$. (Lemma 4.4 of Arai [3]). Then the invertibility of the operator $A_{\lambda,1}$ $(=A_\lambda)$ and the boundedness of A_λ^{-1} from $H^{-1}(R)$ to $H^1(R)$ follow directly. So, $(A_\lambda^{-1}p, p)$ has the meaning for $p \in H^{-1}(R)$ where (,) stands for the duality between $H^1(R)$ and $H^{-1}(R)$. For any $u \in \mathcal{S}(R^3 \times R)$, $\square_F^{-1}u$ belongs to $\mathcal{S}'(R^3\times R)$. So we may regard the term $(\square_F^{-1}u, u)$ as the duality between $\mathcal{S}(R^3\times R)$ and $\mathcal{S}'(R^3\times R)$. Modifying the proof of Lemma 4.3 of Arai [1], we may show easily that the terms $(\rho, \square_F^{-1}u)$ and $(\rho, \square_F^{-1}(\rho p))$ belong to $H^{-1}(R)$ for any $p \in H^{-1}(R)$ and $u \in \mathcal{S}(R^3)$. (See, Bogolubov *et al.* [3] and Gelfand-Shilov [5]).

So the functional $Z(p, u)$ defined by $(1.9)_1$ is well defined on $H^{-1}(R) \times \mathcal{S}(R^3\times R)$. Calculating Gateaux differentials of Z, we may show easily that Z satisfies $(1.1)_1$ with $(1.2)_1$.

For any N, we may proceed analogously.

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