108. Fourier Transforms of Nilpotently Supported Invariant Functions on a Finite Simple Lie Algebra*)

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0. Let $\mathfrak{G}$ be a connected simple algebraic group defined over a finite field $k=F_q$, and let $\mathfrak{g}=\mathrm{Lie}(\mathfrak{G})$, the Lie algebra of $\mathfrak{G}$. We denote by σ the Frobenius morphism, and by G (resp. g) the set $\mathfrak{G}_\sigma$ (resp. $\mathfrak{g}_\sigma$) of σ-fixed points of $\mathfrak{G}$ (resp. $\mathfrak{g}$). Let $\mathrm{Inv}(g)$ be the space of C-valued $\mathrm{Ad}(G)$-invariant functions on g and $\mathrm{Inv}(g_0)$ the subspace of $\mathrm{Inv}(g)$ consisting of all $f\in\mathrm{Inv}(g)$ supported by the set g_0 of nilpotent elements of g. In § 2, we introduce an operation $f\to f^\wedge$ for $f\in\mathrm{Inv}(g)$, and in § 3, we define the 'Fourier transform' $\mathcal{F}(f)$ for $f\in\mathrm{Inv}(g_0)$. The main result (Theorem 3) of this paper says that these two operations coincide with each other on a relatively large subspace $\mathrm{Inv}(g_0)'$ of $\mathrm{Inv}(g_0)$, if the characteristic of k is not too small. As a corollary, we can prove orthogonality relations (Cor. 2) for $\{\mathcal{F}(1_{O_\sigma})\}_O$, where O runs over the set of σ-stable nilpotent $\mathrm{Ad}(\mathfrak{G})$-orbits in $\mathfrak{g}$ and 1_{O_σ} is the characteristic function of O_σ. This can be considered as a counterpart to a result [7, 5.6] of T. A. Springer. (He treated the case of strongly regular (semisimple) orbits rather than nilpotent orbits.) At the end of the paper we present a curious fact (Theorem 4) on the distribution of nilpotent elements in g. Although this result is not directly related to our main results, Theorem 4 and Corollaries 1, 2 show that the variety $\mathfrak{g}_0$ of nilpotent elements of $\mathfrak{g}$ sometimes looks like a $2N$-dimensional vector subspace of $\mathfrak{g}$, where $2N=\dim\mathfrak{g}_0$.

Details and proofs are omitted and will be published elsewhere.

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1. Let $\mathfrak{B}$ be a σ-stable Borel subgroup of $\mathfrak{G}$ and $\mathfrak{T}$ a σ-stable maximal torus contained in $\mathfrak{B}$. Put $B=\mathfrak{B}_\sigma$ and $N(\mathfrak{T})=$the normalizer of $\mathfrak{T}$ in $\mathfrak{G}$. Then $(G,B,N(\mathfrak{T})_\sigma)$ is a Tits system with the Weyl group $W=N(\mathfrak{T})_\sigma/\mathfrak{T}_\sigma$. Let (W,R) be the associated Coxeter system. Then, to each $J\subset R$, there corresponds a σ-stable parabolic subgroup $\mathfrak{P}_J$ of

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$\mathfrak{G}$ containing $\mathfrak{B}$. Let $\mathfrak{V}_J$ be the unipotent radical of $\mathfrak{P}_J$. Put $P_J=(\mathfrak{P}_J)_\sigma$, $V_J=(\mathfrak{V}_J)_\sigma$, $p_J=\mathrm{Lie}(\mathfrak{P}_J)_\sigma$ and $v_J=\mathrm{Lie}(\mathfrak{V}_J)_\sigma$.

If $\mathfrak{H}$ is a σ-stable algebraic subgroup of $\mathfrak{G}$, $H=\mathfrak{H}_\sigma$ and $h=\mathrm{Lie}(\mathfrak{H})_\sigma$, we denote by $\mathrm{Inv}(H)$ (resp. $\mathrm{Inv}(h)$) the space of C-valued class functions on H (resp. $\mathrm{Ad}(H)$-invariant functions on h) with the inner product

$$\langle f_1, f_2\rangle_H=|H|^{-1}\sum_{x\in H} f_1(x)\overline{f_2(x)} \qquad (f_i\in \mathrm{Inv}(H))$$

$$(\text{resp. } \langle f_1, f_2\rangle_h=|H|^{-1}\sum_{x\in h} f_1(x)\overline{f_2(x)} \qquad (f_i\in \mathrm{Inv}(h))).$$

Imitating the definition of the inducing map $\mathrm{ind}_H^G: \mathrm{Inv}(H)\to\mathrm{Inv}(G)$, we define the inducing map $\mathrm{ind}_h^g: \mathrm{Inv}(h)\to\mathrm{Inv}(g)$ by

$$\mathrm{ind}_h^g(f)(A)=|H|^{-1}\sum_{x\in G,\ \mathrm{Ad}(x)A\in h} f(\mathrm{Ad}(x)A)$$

for $f\in\mathrm{Inv}(h)$ and $A\in g$.

2. Let J be a subset of R. For $f\in\mathrm{Inv}(G)$ (resp. $f\in\mathrm{Inv}(g)$), we define an element f_J of $\mathrm{Inv}(P_J)$ (resp. $\mathrm{Inv}(p_J)$) by

$$f_J(x)=|V_J|^{-1}\sum_{y\in V_J} f(xy) \qquad (x\in P_J)$$

$$(\text{resp. } f_J(X)=|V_J|^{-1}\sum_{Y\in v_J} f(X+Y) \qquad (X\in p_J)).$$

Then

$$f^\wedge=\sum_{J\subset R}(-1)^{|J|}\,\mathrm{ind}_{P_J}^G(f_J)$$

$$(\text{resp. } f^\wedge=\sum_{J\subset R}(-1)^{|J|}\,\mathrm{ind}_{p_J}^g(f_J))$$

is again an element of $\mathrm{Inv}(G)$ (resp. $\mathrm{Inv}(g)$).

Theorem 1. (i) *$(f^\wedge)^\wedge=f$ for any $f\in\mathrm{Inv}(G)$ (resp. $\mathrm{Inv}(g)$).*

(ii) *$\langle f_1, f_2\rangle_G=\langle \hat{f_1}, \hat{f_2}\rangle_G$ (resp. $\langle f_1, f_2\rangle_g=\langle \hat{f_1}, \hat{f_2}\rangle_g$) for any $f_1, f_2\in\mathrm{Inv}(G)$ (resp. $\mathrm{Inv}(g)$).*

(iii) *Suppose that f is an irreducible character of G. Then $f^\wedge$ or $-f^\wedge$ is an irreducible character.*

Remark. This has also been proved by D. Alvis [1] independently. See also Curtis [2] and Deligne-Lusztig [3].

3. From now on we need the following:

Assumption 1. The characteristic p of k is good ([9, p. 178]) for $\mathfrak{G}$. If $\mathfrak{G}$ is of type A_l and p devides $l+1$, we also assume that $\mathfrak{G}$ is simply connected, i.e., $\mathfrak{G}\cong SL_n$ over $\bar{k}$.

Let $\kappa(\ ,\ )$ be a symmetric, $\mathrm{Ad}(\mathfrak{G})$-invariant bilinear form on $\mathfrak{g}$ defined over k. If $\mathfrak{G}$ is not of type A_l, we take $\kappa(\ ,\ )$ to be non-degenerate. If $\mathfrak{G}$ is of type A_l, we put

$$\kappa(X, Y)=\mathrm{Trace}\, XY \qquad (X, Y\in\mathfrak{g}=sl_n).$$

(See [9, p. 184].)

Let g_0 and $\mathrm{Inv}(g_0)$ be as in § 0. For $f\in\mathrm{Inv}(g_0)$, the (modified) Fourier transform $\mathcal{F}(f)$ ($\in\mathrm{Inv}(g_0)$) is defined by

$$\mathcal{F}(f)(X)=\begin{cases} q^{-N}\sum_{Y\in g_0}\chi(\kappa(X^*, Y))f(Y) & (X\in g_0);\\ 0 & (X\in g\setminus g_0),\end{cases}$$

where χ is a non-trivial additive character of k, $X\to X^*$ is an opposition automorphism of $\mathfrak{g}$ (which acts as -1 on the root system of $\mathfrak{g}$) and $N=1/2(\dim\mathfrak{g}_0)=$ the number of positive roots of $\mathfrak{G}$.

Remark. Usually (see e.g. [8]) the Fourier transform $F(f)$ of a function f on g is defined by

$$F(f)(X)=q^{-1/2\,(\dim g)}\sum_{Y\in g}\chi(\kappa(X,Y))f(Y)\qquad (X\in g).$$

4. Let $\mathfrak{N}$ be a σ-stable subgroup of $\mathfrak{B}_\phi$ (=the unipotent radical of $\mathfrak{B}$) normalized by $\mathfrak{B}$, and let $n=\mathrm{Lie}(\mathfrak{N})_\sigma$. We denote by $\mathrm{Inv}(g_0)'$ the subspace of $\mathrm{Inv}(g_0)$ spanned by all elements of the form $\mathrm{ind}_n^g(1_n)$ for various $\mathfrak{N}$. In the proofs of Theorems 2 and 3 below we use a classification theorem of nilpotent orbits due to Dynkin [4], Kostant [6] and Springer-Steinberg [9]. The following assumption is made just for this reason.

Assumption 2. If $\mathfrak{G}$ is of type E_6, E_7, E_8, F_4 or G_2, we assume that $p\geqq 4m+3$, where m is the height of the highest root of $\mathfrak{G}$. (If G is of type A_l, B_l, C_l or D_l, Assumption 1 above is already sufficient.)

Remark. It is almost certain that the restrictions on p for exceptional groups are too strong.

Theorem 2. *For $A\in g_0$, we denote by $O(A)$ the $\mathrm{Ad}(\mathfrak{G})$-orbit of A, and by $1_{O(A)_\sigma}$ the characteristic function of $O(A)_\sigma$.*

(i) *Let $f\in\mathrm{Inv}(g_0)'$ and $A\in g_0$. Then $f\cdot 1_{O(A)_\sigma}\in\mathrm{Inv}(g_0)'$.*

(ii) *$1_{O(A)_\sigma}\in\mathrm{Inv}(g_0)'$ for any $A\in g_0$.*

5. **Theorem 3.** *$f^\wedge=\mathcal{F}(f)$ for any $f\in\mathrm{Inv}(g_0)'$.*

Remark. As can be easily seen from the case that $\mathfrak{G}=SL_2$ and $p\neq 2$, one can not replace $\mathrm{Inv}(g_0)'$ with $\mathrm{Inv}(g_0)$ in Theorem 3.

Combining Theorems 1 and 3, we get:

Corollary 1. (i) *$\mathcal{F}(\mathcal{F}(f))=f$ for any $f\in\mathrm{Inv}(g_0)'$.*

(ii) *$\langle f_1,f_2\rangle_g=\langle\mathcal{F}(f_1),\mathcal{F}(f_2)\rangle_g$ for any $f_1,f_2\in\mathrm{Inv}(g_0)'$.*

By Theorem 2 (ii), we have the following orthogonality relations as a special case of Corollary 1 (ii).

Corollary 2. *Let $A, A'\in g_0$. Then*

$$\sum_{X\in g_0}\mathcal{F}(1_{O(A)_\sigma})(X)\mathcal{F}(1_{O(A')_\sigma})(X)=\begin{cases}|O(A)_\sigma| & \text{if } O(A)=O(A');\\ 0 & \text{otherwise.}\end{cases}$$

6. The next result can be proved under the Assumption 1.

Theorem 4. *Let $b=\mathrm{Lie}(\mathfrak{B})_\sigma$ and X be an arbitrary element of g. Then the number of nilpotent elements in the set $b+X$ is always q^N.*

Remark. Compare with the author's previous result [5, Theorems 7.2, 7.5] on the distribution of regular unipotent elements in G.

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