107. On Fourier Coefficients of Klingen's Eisenstein Series of Degree Three

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In this note, we shall give explicit formulas for some Fourier coefficients of Klingen's Eisenstein series of degree three. The author would like to thank Prof. Kitaoka for his helpful advices.

Let H_n denote the Siegel upper half-space of $n\times n$ symmetric complex matrices with positive definite imaginary part. Put $\Gamma_n=\mathrm{Sp}(n,\boldsymbol{Z})$. Let M_n^k denote the space of Siegel modular forms of degree n and of weight k with respect to Γ_n. Let $f\in M_r^k$ be a cusp form on H_r. For even k, $k>n+r+1$ and $n>r$, the Eisenstein series $E_{n,r}^k(Z,f)$ is defined, following Klingen [5], by the following series:

$$E_{n,r}^k(Z,f)=\sum_{M\in\Delta_{n,r}\backslash\Gamma_n} f(M\langle Z\rangle^*)|CZ+D|^{-k}$$

and its Fourier expansion is denoted by

$$=\sum_{T\geq 0} a(T;[f])\exp(2\pi i\,\mathrm{Tr}(TZ)).$$

In [3], Harris proved that $E_{n,r}^k(Z,f)$ has algebraic Fourier coefficients whenever f does without showing any explicit formula for $a(T;[f])$ (see also [6]). On the other hand, Mizumoto [8] proved explicit formulas for some Fourier coefficients of $E_{2,1}^k(Z,f)$. Here we shall study Fourier coefficients of $E_{3,2}^k(Z,f)$ using a different idea from Mizumoto's, suggested by Kitaoka.

To state our result precisely, let $f\in M_2^k$ be a cusp form which is a common eigenfunction for all Hecke operators $T(m)$. We denote its Fourier expansion by

$$f(z)=\sum_{N\geq 0} b(N)\exp(2\pi i\,\mathrm{Tr}(NZ)),\qquad Z\in H_2.$$

Let $Z_f^{(2)}(s)$ denote the symmetric square of the zeta-function corresponding to f (see Def. 2.1. in [1]). Let T be any 3×3 positive definite semi-integral matrix. Put $\Delta(T)=|2T|$. For such T, we associate an analytic class invariant $\vartheta_T(Z)$, $Z\in H_2$, by the following series:

$$\vartheta_T(Z)=\sum_{M\in M_{2\times 3}(Z)}\exp(2\pi i\,\mathrm{Tr}(MT^tMZ)).$$
$$=\sum_{N\geq 0} c(N)\exp(2\pi i\,\mathrm{Tr}(NZ)).$$

For any Dirichlet character χ, we define a Dirichlet series $D(s,f,\vartheta_T,\chi)$ by the following series like in [7]:

$$D(s,f,\vartheta_T,\chi)=\sum_{\{N\}}\frac{b(N)c(N)\chi(|2N|)}{\varepsilon(N)|N|^s}.$$

Since $k \geqq 10$, it can be proved that $D(s, f, \vartheta_T, \chi)$ converges at $s=k-3/2$. Then we can prove

Theorem. *The notation being as above, we suppose $\varDelta(T)/2$ is odd square free. Then we have*

$a(T;[f])$

$$=(-1)^{k/2}2^{-1}\frac{(2\pi)^{k-1}}{(k-2)!}|T|^{k-2}\frac{\zeta(2k-4)\sum_{0<d|\varDelta(T)/2} D(k-3/2, f, \vartheta_T, \chi_d)d^{2-k}}{Z_f^{(2)}(2k-3)},$$

where Dirichlet characters $\chi_d(m)$ are defined by $\chi_d(m)=(m/d)$, the Jacobi symbol.

Sketch of the proof. Main ingredients of the proof are (1) Kitaoka's formula for $a(T;[f])$ in [4] (Theorem at p. 113) and (2) Andrianov's formula which gives the relation between $Z_f^{(2)}(s)$ and the series of the form

$$\sum_{M \in SL(2,Z)\backslash S^1} b(MN^tM)|M|^{-s}$$

where we use the same notation as in [1] and [4] without any reference. The proof proceeds as follows; Theorem 2.1 in [1] implies

$$\frac{\zeta(2k-4)D(k-3/2, f, \vartheta_T, \chi_1)}{Z_f^{(2)}(2k-3)}$$
$$=\sum_{\{\{M\}\}} L(k-1, \chi_{MT^tM})^{-1}\Phi_f(2k-3, MT^tM)|MT^tM|^{k-3/2}$$

where $M \in M_{2\times 3}(Z)$ ranges over representatives of all equivalence classes of primitive elements: for primitive elements $M, M' \in M_{2\times 3}(Z)$, $M \sim M'$ means $M=VM'$ for some $V \in GL(2, Z)$. Then it is easily seen that there exists a one to one correspondence between $\{\{M\}\}$ and $\{{}^tU \in P_{3,2}\backslash GL(2, Z)\}$ such that $MT^tM=(U^{-1}T^tU^{-1})_1$. Therefore the proof is done when we calculate coefficients of $b(({}^tC^{-1}U^{-1}T^tU^{-1}C^{-1})_1)|(U^{-1}T^tU^{-1})_1|^{3/2-k}|T|^{k-2}$ in Kitaoka's formula which are infinite sums with respect to (C_4, D_3, D_4) and show that these are something like $L(k-1, \chi_{MT^tM})^{-1}\mu(\delta)$ for $\delta=(C_4, D_4)$, and μ the Möbius function.

Remark 1. The condition about T is only for the convenience of calculations.

Remark 2. When we want to generalize our theorem for arbitrary n and r, we need a formula analogous to Theorems 2, 1 in [1] for the trivial character (cf. [2]).

Remark 3. After this work had been done, Kurokawa informed us that he suggested a similar formula for $a(T:[f])$ to ours in [6].

References

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