106. On the Attractivity Properties for the Equation $x''+a(t)f_1(x)g_1(x')x'+b(t)f_2(x)g_2(x')x=e(t,x,x')$

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1. Introduction. In this paper we shall study the asymptotic behavior of solutions of the second order differential equation

(1) $$x''+a(t)f_1(x)g_1(x')x'+b(t)f_2(x)g_2(x')x=e(t,x,x')$$

or an equivalent system

(2) $$x'=y, \qquad y'=-a(t)f_1(x)g_1(y)y-b(t)f_2(x)g_2(y)x+e(t,x,y),$$

where $a(t)>0$, $b(t)>0$, $f_i(x)>0$ and $g_i(y)>0$ $(i=1,2)$.

In [1], the following theorem was given by T. A. Burton for the system

(3) $$x'=y, \qquad y'=-p(x)\,|y|^{\alpha}y-g(x),$$

where $p(x)>0$ and $0\leqq\alpha<1$.

Theorem (Burton). *The zero solution of* (3) *is globally asymptotically stable if and only if* $\int_0^{\pm\infty}[p(x)+|g(x)|]dx=\pm\infty$.

In [2], Burton had an extension of this theorem for the following system:

(4) $$x'=y, \qquad y'=-f(x)h(y)y-g(x)+e(t).$$

On the other hand, for the system

(5) $$x'=y, \qquad y'=-f(x)h(y)y-g(x)k(y)+e(t),$$

J. W. Heidel proved in [3] that if $\int_0^{\pm\infty}[f(x)+|g(x)|]dx=\pm\infty$ and if $k(y)$ satisfies some conditions, then all solutions of (5) converge to the origin as $t\to\infty$, that is the origin is attractive for (5).

The purpose of this paper is to give a sufficient condition and a necessary condition for the convergence of all solutions of (2) to the origin as $t\to\infty$ under the following assumptions.

(I) *$a(t)$ and $b(t)$ are continuously differentiable in* $[0,\infty)$.

(II) *$f_1(x)$, $f_2(x)$, $g_1(y)$ and $g_2(y)$ are continuous in R^1 and $e(t,x,y)$ is continuous in* $[0,\infty)\times R^2$.

(III) $\displaystyle\int_0^{\infty}\frac{|a'(t)|}{a(t)}dt<\infty$ *and* $\displaystyle\int_0^{\infty}\frac{|b'(t)|}{b(t)}dt<\infty$.

(IV) $\displaystyle\int_0^{y}\frac{v}{g_2(v)}dv\to\infty$ *as* $|y|\to\infty$.

(V) $\displaystyle\frac{y^2}{g_2(y)}\leqq M\int_0^{y}\frac{v}{g_2(v)}dv$ *for $y\in R^1$, where $M>0$.*

(VI) *There exist continuous, nonnegative functions* $r_1(t)$ *and* $r_2(t)$ *such that*

$$|e(t, x, y)| \leqq r_1(t) + r_2(t)|y|^l, \quad 0 \leqq l \leqq 1, \quad \int_0^\infty r_i(t)dt < \infty \quad (i=1, 2).$$

2. Lemmas, theorems and their proofs. We give the following lemmas without their proofs. (See [4], [5].)

Lemma 1. *If the function* $a(t)$ *satisfies* (I) *and* (III), *then there exist constants* a_1 *and* a_2 *such that* $0 < a_1 \leqq a(t) \leqq a_2$ *for* $t \geqq 0$.

Lemma 2. *Suppose the assumptions* (I)–(III) *and* (VI). *Then every bounded solution of* (2) *converges to the origin* (0, 0) *as* $t \to \infty$.

It is convenient to define the functions F_1, F_2, G_1, G_2 and G_L by

$$F_1(x) = \int_0^x f_1(u)du, \quad F_2(x) = \int_0^x uf_2(u)du, \quad G_1(y) = \int_0^y \frac{1}{g_1(v)}dv,$$

$$G_2(y) = \int_0^y \frac{v}{g_2(v)}dv \quad \text{and} \quad G_L(y) = LG_2(y) - \frac{1}{2}[G_1(y)]^2, \qquad \text{where } L > 0.$$

Theorem 1. *Suppose the assumptions* (I)–(VI). *If* $\int_0^{\pm\infty} \{f_1(x) + |x| f_2(x)\}dx = \pm\infty$, *then every solution of* (2) *converges to the origin* (0, 0) *as* $t \to \infty$, *that is the origin is attractive.*

Proof. It follows from (II) and (V) that $|y|^{1+l}/g_2(y) \leqq m + MG_2(y)$ for $y \in R^1$, $0 \leqq l \leqq 1$, where $m > 0$. Let $(x(t), y(t))$ be a solution of (2) through (t_0, x_0, y_0). Let $V_1(t, x, y) = b(t)F_2(x) + G_2(y) + m/M$. Differentiating $V_1(t) = V_1(t, x(t), y(t))$ with respect to t, we have

$$\begin{aligned} V_1'(t) &\leqq |b'(t)| F_2(x) + r_1(t)\frac{|y|}{g_2(y)} + r_2(t)\frac{|y|^{1+l}}{g_2(y)} \\ &\leqq \left\{\frac{|b'(t)|}{b(t)} + Mr_1(t) + Mr_2(t)\right\}V_1(t) \qquad \text{for } t \geqq t_0. \end{aligned}$$

Integrating $V_1'(t)$ from t_0 to t and applying Gronwall's lemma, we obtain

$$(7) \qquad V_1(t) \leqq V_1(t_0) \exp\left[\int_0^\infty \left\{\frac{|b'(s)|}{b(s)} + Mr_1(s) + Mr_2(s)\right\}ds\right] = L_1,$$

and $G_2(y(t)) \leqq V_1(t) \leqq L_1$ for $t \in [t_0, t_1)$, whenever the solution $(x(t), y(t))$ is defined in $[t_0, t_1)$. Hence the boundedness of $y(t)$ follows from (IV). This implies that the solution $(x(t), y(t))$ is defined in the future, since $x'(t) = y(t)$. And so there exists $B > 0$ such that $|y(t)| \leqq B$ for $t \geqq t_0$. Then in the case that $F_2(x) \to \infty$ as $x \to \pm\infty$, it follows from (III) and (7) that $F_2(x(t)) \leqq b_1^{-1}L_1$ for $t \geqq t_0$. Therefore $x(t)$ is bounded for $t \geqq t_0$. On the other hand, in the case that $F_1(x) \to \pm\infty$ as $x \to \pm\infty$, we define the function

$$V_2(t, x, y) = \begin{cases} \frac{1}{2}[a(t)F_1(x) + G_1(y) + G_0]^2 + 1 & \text{for } t \geqq 0, \quad x \geqq 1, \quad |y| \leqq B \\ \frac{1}{2}[a(t)F_1(x) + G_1(y) - G_0]^2 + 1 & \text{for } t \geqq 0, \quad x \leqq -1, \quad |y| \leqq B, \end{cases}$$

where $G_0 > \sup_{|y| \leqq B} |G_1(y)|$. Now suppose that $x(t) \geqq 1$ for $t \in [t_1, t_2]$. Differentiating $V_2(t) = V_2(t, x(t), y(t))$ with respect to t, we have

$$V_2'(t) \leqq [a(t)F_1(x) + G_1(y) + G_0]\left[|a'(t)|\, F_1(x) + \frac{|e(t, x, y)|}{g_1(y)}\right]$$
$$\leqq \sqrt{2V_2(t)}\left[\frac{|a'(t)|}{a(t)}\sqrt{2V_2(t)} + \{r_1(t) + B^\ell r_2(t)\}\left\{\inf_{|y| \leqq B} g_1(y)\right\}^{-1}\right]$$
$$\leqq L_2\left[\frac{|a'(t)|}{a(t)} + r_1(t) + r_2(t)\right]V_2(t) \qquad \text{for } t \in [t_1, t_2]$$

where $L_2 > 0$. Then it is easily shown that $V_2(t) \leqq L_3 V_2(t_1)$ and hence $F_1(x(t)) \leqq a_1^{-1}\sqrt{2L_3V_2(t_1)}$ for $t \in [t_1, t_2]$, where L_3 is independent of t_1 and t_2. Since $F_1(x) \to \infty$ as $x \to \infty$, there exists a constant $\bar{x} > 1$ such that $x(t) \leqq \bar{x}$ for $t \in [t_1, t_2]$. If $x(t_1) = 1$, then $V_2(t_1) \leqq 1/2[a_2F_1(1) + 2G_0]^2 + 1$. On the other hand, if $x_0 \geqq 1$ and if $t_1 = t_0$, then $V_2(t_1) \leqq 1/2[a_2F_1(x_0) + 2G_0]^2 + 1$. Hence $\bar{x}$ is independent of t_1 and t_2. Therefore $x(t)$ is bounded from above for $t \geqq t_0$. Similarly, the boundedness from below of $x(t)$ follows by using $V_2(t, x, y)$.

In the case that $F_1(x) \to \infty$ as $x \to \infty$ and $F_2(x) \to \infty$ as $x \to -\infty$ or in the case that $F_2(x) \to \infty$ as $x \to \infty$ and $F_1(x) \to -\infty$ as $x \to -\infty$, using the functions $V_1(t, x, y)$ and $V_2(t, x, y)$, we can show the boundedness of $x(t)$. Thus every solution of (2) is bounded. This implies from Lemma 2 that every solution of (2) converges to $(0, 0)$ as $t \to \infty$. Q.E.D.

Theorem 2. *Suppose the assumptions* (I)–(III) *and* (VI). *If every solution of* (2) *converges to* $(0, 0)$ *as* $t \to \infty$, *then* $\int_0^{\pm\infty} \{f_1(x) + |x| f_2(x)\} dx = \pm\infty$.

Proof. We shall prove only that $\int_0^\infty \{f_1(x) + x f_2(x)\} dx = \infty$. Suppose $\int_0^\infty \{f_1(x) + x f_2(x)\} dx < \infty$. Let $V_3(y) = \int_0^y (1/(1+|v|)) dv$. Then there exists $y_0 > 1$ such that $V_3(y_0) > V_3(1) + 1 + \int_0^\infty \{r_1(t) + r_2(t)\} dt$, because $V_3(y) \to \pm\infty$ as $y \to \pm\infty$. Let $g^* = \sup_{1 \leqq y \leqq y_0} \{g_1(y) + g_2(y)/y\}$ and choose x_0 so large that $(a_2 + b_2) g^* \int_{x_0}^\infty \{f_1(x) + x f_2(x)\} dx < 1$. Let $(x(t), y(t))$ be a solution of (2) through (t_0, x_0, y_0). Since $y(t)$ converges to zero as $t \to \infty$, we can find two numbers $t_1 \geqq t_0$ and $t_2 > t_1$ such that $y(t_1) = y_0$, $y(t_2) = 1$, $y(t) \geqq y_0$ for $t \in (t_0, t_1)$ and $1 < y(t) < y_0$ for $t \in (t_1, t_2)$. Then $x(t) > x_0$ for $t \in [t_0, t_2]$. Differentiating $v(t) = V_3(y(t))$ with respect to t, we obtain from (VI), for $t \in [t_1, t_2]$

$$v'(t) \geqq -a_2 g^* f_1(x) x' - b_2 g^* f_2(x) x x' - r_1(t) - r_2(t).$$

Hence

$$v(t_2) \geqq v(t_1) - a_2 g^* \int_{x_0}^{x(t_2)} f_1(x) dx - b_2 g^* \int_{x_0}^{x(t_2)} x f_2(x) dx$$

$$-\int_{t_1}^{t_2}\{r_1(t)+r_2(t)\}dt$$

$$\geqq V_3(y_0)-(a_2+b_2)g^*\int_{x_0}^{\infty}\{f_1(x)+xf_2(x)\}dx-\int_0^{\infty}\{r_1(t)+r_2(t)\}dt$$

$$>V_3(y_0)-1-\int_0^{\infty}\{r_1(t)+r_2(t)\}dt.$$

Then we have $v(t_2)>V_3(1)=v(t_2)$, which is a contradiction. Thus we conclude that $\int_0^{\infty}\{f_1(x)+xf_2(x)\}dx=\infty$. Q.E.D.

Now the following Theorem 3 is an immediate consequence of Theorems 1 and 2.

Theorem 3. *Suppose the assumptions* (I)–(VI). *Then every solution of* (2) *converges to the origin* (0, 0) *as* $t\to\infty$ *if and only if*

$$\int_0^{\pm\infty}\{f_1(x)+|x|f_2(x)\}dx=\pm\infty.$$

Remark. If $e(t,x,y)\equiv 0$, then the system (2) has the zero solution $(x(t), y(t))=(0,0)$. In this case, Theorem 3 implies that the zero solution is globally asymptotically stable if and only if $\int_0^{\pm\infty}\{f_1(x)+|x|f_2(x)\}dx=\pm\infty$ under the assumptions (I)–(VI).

References

[1] T. A. Burton: The generalized Lienard equation. SIAM J. Control, **3**, 223–230 (1965).

[2] ——: On the equation $x''+f(x)h(x')x'+g(x)=e(t)$. Ann. Math. Pura Appl., **85**, 277–285 (1970).

[3] J. W. Heidel: A Liapunov function for a generalized Lienard equation. J. Math. Anal. Appl., **39**, 192–197 (1972).

[4] S. Sakata and M. Yamamoto: On the boundedness and the attractivity properties of nonlinear second order differential equations. Proc. Japan Acad., **57A**, 111–116 (1981).

[5] M. Yamamoto and S. Sakata: On the boundedness of solutions and the attractivity properties for nonlinear second-order differential equations. Math. Japon. (to be published).